

# Topological interpretation of link invariants from finite quandles

Takefumi Nosaka

## Abstract

This paper demonstrates a topological meaning of the quandle cocycle invariants of links with respect to finite connected quandles  $X$ , from a perspective of homotopy theory: Precisely, for any prime  $\ell$  which does not divide the type of  $X$ , the  $\ell$ -torsion of this invariants is reduced to be a sum of the colouring polynomial and a  $\mathbb{Z}$ -equivariant part of the Dijkgraaf-Witten invariant of a cyclic branched covering space. Furthermore, our homotopical approach involves application of computing some third homologies and second homotopy groups of the classifying spaces of quandles, from group cohomology.

**Keywords** Quandle, group homology, homotopy group, link, branched covering, bordism group, orthogonal and symplectic group, mapping class group

## 1 Introduction

A quandle,  $X$ , is a set with a binary operation  $\triangleleft : X^2 \rightarrow X$  the axioms of which were motivated by knot theory. Actually, for an oriented link  $L$  in the 3-sphere  $S^3$ , we can define the link quandle  $Q_L$  [Joy], which is, roughly speaking, the conjugacy classes of  $\pi_1(S^3 \setminus L)$  including the meridians; A quandle homomorphism  $\mathcal{C} : Q_L \rightarrow X$  is called an  $X$ -coloring of  $L$ . Fenn, Rourke and Sanderson [FRS1, FRS2, RS] introduced a space  $BX$  as an analogy of the classifying spaces of groups; they further defined a fundamental class in  $\pi_2(BQ_L)$  diagrammatically, and consider its push-forward,  $\Xi_X(L; \mathcal{C}) \in \pi_2(BX)$ , by  $X$ -colored links  $(L, \mathcal{C})$ ; Then, when  $X$  is of finite order, the quandle homotopy invariant  $\Xi_X(L) \in \mathbb{Z}[\pi_2(BX)]$  of a link  $L$  is defined as the formal sum of  $\Xi_X(L; \mathcal{C})$  running over all  $X$ -colorings  $\mathcal{C}$  of  $L$ , i.e.,  $\Xi_X(L) = \sum_{\mathcal{C}} \Xi_X(L; \mathcal{C})$ . The author [N1, N2] studied quantitatively  $\pi_2(BX)$  and the invariant  $\Xi_X(L)$  for some quandles  $X$ .

From a cohomological viewpoint, Carter *et.al* [CJKLS, CEGS, CKS] introduced quandle cocycle invariants  $\Phi_\phi(L) \in \mathbb{Z}[A]$  of links constructed from 2-cocycles  $\phi$  of  $X$  with a local coefficient group  $A$ . The invariants were much studied, since it can be relatively computed, compared with the homotopy group  $\pi_2(BX)$ . As is known [RS], any such cocycle invariants can be obtained from the quandle homotopy invariant via the Hurewicz map  $\pi_2(BX) \rightarrow H_2(BX; A)$  with local system (see [N1, §2] for the detailed formula).

However, these invariants of links were constructed from link diagrams combinatorially. So, two papers [Kab, HN] tried to give a topological meaning of these invariant; however, the second homology  $H_2(BX; \mathbb{Z})$  is an obstacle to study the space  $BX$ , and accordingly successful works are only for the simplest quandle of the form  $X = \mathbb{Z}/(2m-1)$  with  $x \triangleleft y := 2y - x$ . In fact, so far, there are only a few topological studies of  $BX$  with respect to general quandles  $X$ . (However, we refer to [FRS2, Cla1, N2] as some topological approaches to  $BX$ )

In this paper we demonstrate a topological meaning of these invariants from, more generally, “connected” quandles, together with computing the homotopy groups  $\pi_2(BX)$ . For this, we shall focus on the quandle *homotopy* invariant  $\Xi_X(L)$ , since the  $\Xi_X(L)$  is universal among

all the cocycle invariants as mentioned above. To state our main theorem, we fix two simple terminologies: A quandle  $X$  is said to be *connected*, if any  $x, y \in X$  admit some  $a_1, \dots, a_n \in X$  such that  $(\cdots(x \triangleleft a_1) \triangleleft \cdots) \triangleleft a_n = y$ ; The type  $t_X$  of  $X$  is the minimal  $N$  such that  $x = (\cdots(x \triangleleft y) \triangleleft \cdots) \triangleleft y$  [ $N$ -times on the right with  $y$ ] for any  $x, y \in X$ .

**Theorem 1.1** (Theorem 3.4<sup>1</sup>). *Let  $X$  be a connected quandle of type  $t_X$ , and of finite order. Let  $\mathcal{H}_X : \pi_2(BX) \rightarrow H_2(BX; \mathbb{Z})$  be the Hurewicz map. Then there is a homomorphism  $\Theta_X$  from  $\pi_2(BX)$  to the third group homology  $H_3^{\text{gr}}(\pi_1(BX); \mathbb{Z})$  such that the sum*

$$\mathcal{H}_X \oplus \Theta_X : \pi_2(BX) \longrightarrow H_2(BX; \mathbb{Z}) \oplus H_3^{\text{gr}}(\pi_1(BX); \mathbb{Z})$$

*is an isomorphism after localization at  $\ell$ , where  $\ell$  is relatively prime to  $t_X$ .*

Furthermore, we will show (Corollary 3.5) that, for any link  $L \subset S^3$ , the homotopy invariant  $\Xi_X(L) \in \mathbb{Z}[\pi_2(BX)]$  is sent to a sum of two invariants via the map  $\mathcal{H}_X \oplus \Theta_X$ : the original quandle cycle invariant  $\in \mathbb{Z}[H_2(BX)]$  in [CJKLS] and “a  $\mathbb{Z}$ -equivariant of Dijkgraaf-Witten invariant of  $\widehat{C}_L^{t_X}$ ”, where  $\widehat{C}_L^{t_X}$  is the  $t_X$ -fold cyclic covering space of  $S^3$  branched over the link  $L$ . Denote the two invariants by  $\Phi_X(L)$  and  $\text{DW}_{\text{As}(X)}^{\mathbb{Z}}(\widehat{C}_L^{t_X})$ , respectively. Formally, this result is summarized to an equality

$$(\mathcal{H}_X \oplus \Theta_X)(\Xi_X(L)) = \Phi_X(L) + \text{DW}_{\text{As}(X)}^{\mathbb{Z}}(\widehat{C}_L^{t_X}) \quad \text{modulo } t_X\text{-torsion.}$$

We here remark that, as is shown [E2], the former invariant  $\Phi_X(L)$  is characterized by longitudes of  $X$ -colored links (see §5 for details): furthermore the latter is roughly defined by a sum of push-forwards of the fundamental class of  $\widehat{C}_L^{t_X}$  via some  $\mathbb{Z}$ -equivariant homomorphisms  $\pi_1(\widehat{C}_L^{t_X}) \rightarrow \pi_1(BX)$  (see (9) for the definition). In conclusion, via the map  $\mathcal{H}_X \oplus \Theta_X$ , the homotopy invariant  $\Xi_X(L)$  without  $t_X$ -torsion is reduced to be the two topological invariants, as desired. As we here emphasise, our theorem indicates that a minimal obstacle in computing  $\pi_2(BX)$  is the type  $t_X$  of  $X$ , rather than the second homology  $H_2(BX)$ ; Moreover, this theorem has observed most parts of  $\pi_2(BX)$  with respect to concrete quandles, while it misses the  $t_X$ -torsions. Actually, there are many quandles whose type are powers of some prime.

So as to compute  $\pi_2(BX)$  in practice, we have to study the image of the map  $\mathcal{H}_X \oplus \Theta_X$  in Theorem 1.1, in particular the fundamental group  $\pi_1(BX)$ . In §A we develop a simple and applicable method for determining  $\pi_1(BX)$  in terms of ‘universal central extensions of groups modulo  $t_X$ -torsion’. Actually, in several cases, we determine  $\pi_1(BX)$  concretely, and compute the homology  $H_3^{\text{gr}}(\pi_1(BX); \mathbb{Z})$ . Furthermore, we can determine the second homologies  $H_2(BX)$  in terms of  $\pi_1(BX)$ , thanks to a method of Eisermann [E2] (see Theorem 5.2 and Appendix B). In summary, we can determined the  $\pi_2(BX)$  exactly without  $t_X$ -torsion from computations of  $H_3^{\text{gr}}(\pi_1(BX))$  and  $H_2(BX)$ .

This paper further investigates and computes some  $t_X$ -torsion of  $\pi_2(BX)$  as well. We will show that homomorphism  $\mathcal{H}_X \oplus \Theta_X$  is an isomorphism for several quandles: precisely, “regular

<sup>1</sup>After §2, we employ reduced notation of  $BX$  for simplicity. For example, we use two groups,  $\text{As}(X)$  and  $\Pi_2(X)$ , such that  $\pi_1(BX) \cong \text{As}(X)$  and  $\pi_2(BX) \cong \mathbb{Z} \oplus \Pi_2(X)$  (see (19)). Furthermore, instead of the homology  $H_*(BX)$ , we mainly deal with the quandle homology  $H_*^Q(X; \mathbb{Z})$  introduced in [CJKLS], and remark two known isomorphisms  $H_2(BX) \cong \mathbb{Z} \oplus H_2^Q(X)$  and  $H_3(BX) \cong \mathbb{Z} \oplus H_2^Q(X) \oplus H_3^Q(X)$  (see (15) and (30)).

Alexander quandles”, most “symplectic quandles over  $\mathbb{F}_q$ ”, and most connected quandles of order  $\leq 8$  (see Theorems 3.10, 3.12). Hence, we have computed the homotopy group  $\pi_2(BX)$ , from which follows computing  $H_2(BX; \mathbb{Z})$  and  $H_3^{\text{gr}}(\pi_1(BX))$  in the right side. For example, regarding the symplectic quandles over  $\mathbb{F}_q$  “in a certain stable range”, the  $\pi_2(BX)$  is isomorphic to  $\mathbb{Z} \oplus K_3(\mathbb{F}_q)$ , where  $K_3(\mathbb{F}_q) \cong \mathbb{Z}/(q^2 - 1)$  is the Quillen  $K$ -group of  $\mathbb{F}_q$ . So, in general, we conjecture that the sum  $\mathcal{H}_X \oplus \Theta_X$  is an isomorphism and plays a good role for many connected quandles  $X$ . Incidentally, as an application to a “Dehn quandle  $\mathcal{D}_g$ ”, which is a certain conjugacy class of the mapping class group and is useful for Lefschetz fibrations, we show that  $\pi_2(B\mathcal{D}_g)$  is either  $\mathbb{Z} \oplus \mathbb{Z}/24$  or  $\mathbb{Z} \oplus \mathbb{Z}/48$  for  $g \geq 7$  (Theorem 8.1).

Our approach to  $\pi_2(BX)$  moreover involves a new method to compute the *third* homology  $H_3(BX)$ , and establish a relation between third *quandle* homologies and *group* homologies. As a general result, with respect to a finite connected quandle  $X$ , we solve some torsion subgroups of the  $H_3(BX)$  in terms of group homology (Theorem 3.13). Furthermore, we compute most torsions of the third homologies  $H_3(BX)$  explicitly of the symplectic quandles and spherical quandles over  $\mathbb{F}_q$  in a stable range (see Theorem 3.17). In addition, letting  $X$  be an Alexander quandle, we will show the isomorphism  $H_3(BX) \cong H_3^{\text{gr}}(\pi_1(BX)) \oplus (H_2(BX) \wedge H_2(BX))$  up to  $2t_X$ -torsion. We here note that most of known methods to compute  $H_3(BX)$  by hand was a result of Mochizuki [Moc] with respect to Alexander quandles of the form  $X = \mathbb{F}_q[T]/(T - \omega)$ , although his presentation of  $H^3(BX; \mathbb{F}_q)$  was not so simple (see [Moc]). However our result implies that the complexity of  $H^3(BX; \mathbb{F}_q)$  stems from that of  $H_3^{\text{gr}}(\pi_1(BX))$ .

Furthermore, we study a close relation between the homologies  $H_3(B\tilde{X})$  and  $\pi_2(B\tilde{X})$  with respect to in a class of “universal quandle coverings  $\tilde{X}$ ”. Such a quandle  $\tilde{X}$  is constructed from a connected quandle  $X$  of  $t_X$  (see §3.3 the definition). As we see Theorem 3.18, if  $X$  is of finite order, our approach above provides isomorphisms

$$\pi_2(B\tilde{X}) \cong H_3(B\tilde{X}) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus \mathbb{Z} \quad \text{up to } t_X\text{-torsion.}$$

This result and viewpoint from extended quandles  $\tilde{X}$  is of vital importance in the proof of Theorem 1.1 (see §6.1) and in a subsequent paper [N4].

Finally, we emphasize two benefits from our study on  $\pi_2(BX)$  in Theorem 1.1. First, our theorem suggests a simple computation of the Dijkgraaf-Witten invariants with respect to a finite group. Although the definition seems very simple (see (8)), it is not so easy to compute this invariants exactly. Actually, most known computations of the invariants are those with respect to abelian groups (see, e.g., [DW, Kab, HN]). However, as mentioned above, the isomorphism  $\Theta_X \oplus \mathcal{H}_X$  implies that we can deal with some  $\mathbb{Z}$ -equivariant parts of this invariants via the quandle cocycle invariants of links. In fact, in the paper [N4] we will compute the invariants of some knots using Alexander quandles  $X$  over  $\mathbb{F}_q$ , whose the  $\pi_1(BX)$  are nilpotent groups (see (36) for the lower central series). As a result, we will calculate some triple Massey products of some Brieskorn manifolds  $\Sigma(n, m, l)$ ; see [N4, §5].

On the other hand, for algebraic topology, our proof of Theorem 1.1 determines the second Postnikov invariant,  $k^3(BX) \in H_{\text{gr}}^3(\pi_1(BX); \pi_2(BX))$ , modulo  $t_X$ -torsion. Indeed, Theorem

1.1 is shown by using the Postnikov tower of  $BX$  written in an exact sequence

$$H_3(BX) \xrightarrow{c_*} H_3^{\text{gr}}(\pi_1(BX)) \xrightarrow{\tau} \pi_2(BX) \xrightarrow{\mathcal{H}_X} H_2(BX) \xrightarrow{c_*} H_2^{\text{gr}}(\pi_1(BX)) \rightarrow 0, \quad (1)$$

where  $\tau$  is the transgression map (see §6.1 for details). To be more precise, we will show (Theorem 6.1) that the both maps  $c_*$  are annihilated by  $t_X$ , and (Proposition 6.13) that the map  $\Theta_X$  in Theorem 1.1 gives a splitting from  $\pi_2(BX)$  [Interestingly, this proposition is proved in a knot-theoretical viewpoint (see §6.4)]. By the definition of  $k^3(BX)$  they therefore conclude Theorem 1.1 and clarify the  $k^3(BX)$  up to  $t_X$ -torsion. Here we compare the exact sequence (1) with some results in [N2], where the  $\pi_2(BX)$  were analyzed with local systems with respect to only “regular” Alexander quandles  $X$ . So, as a consequence of our theorem, we emphasize that the classical Postnikov tower is a powerful method to study  $\pi_2(BX)$  for general finite quandles.

This paper is organized as follows. Section 2 reviews the quandle homotopy invariant. Section 3 states our results. Section 4 constructs the homomorphism  $\Theta_X$ . Section 5 reviews a method to compute the second quandle homology according to [E2]. Section 6 proves the Theorem 1.1. Section 7 contains the proofs of Theorems 3.10, 3.12: precisely, we will compute some  $\pi_2(BX)$  and  $H_3(BX)$  concretely. Section 8 determines  $\pi_2(BX)$  of the Dehn quandle in a stable range. Section 9 discusses the third homology  $H_3(BX)$ . In addition, Appendix A proposes a method to calculate automorphism groups of quandles. Appendix B compute some second quandle homologies.

**Conventional notation** Throughout this paper, most homologies are with (trivial) integral coefficients; so we often omit writing coefficients, e.g.,  $H_n(X)$ . We denote the group homology of a group  $G$  by  $H_n^{\text{gr}}(G)$ . Furthermore, we denote a  $\mathbb{Z}$ -module  $M$  localized at a prime  $\ell$  by  $M_{(\ell)}$ . Moreover, a homomorphism  $f : A \rightarrow B$  between abelian groups is said to be *an isomorphism modulo  $N$ -torsion*, denoted by  $f : A \cong B \pmod{N}$ , if the localization of  $f$  at  $\ell$  is an isomorphism for any prime  $\ell$  that does not divide  $N$ . In addition, we assume that every manifolds are in  $C^\infty$ -class and oriented, and that any fields is not of characteristic 2.

## 2 Review of quandles and the quandle homotopy invariants

To establish our results in §3, we will review quandles in §2.1, and quandle homotopy invariants of links in §2.2.

### 2.1 Review of quandles

A *quandle* is a set,  $X$ , with a binary operation  $\triangleleft : X \times X \rightarrow X$  such that

- The identity  $a \triangleleft a = a$  holds for any  $a \in X$ .
- The map  $(\bullet \triangleleft a) : X \rightarrow X$  defined by  $x \mapsto x \triangleleft a$  is bijective for any  $a \in X$ .
- The identity  $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$  holds for any  $a, b, c \in X$ .

A quandle  $X$  is said to be *of type*  $t_X$ , if  $t_X > 0$  is the minimal  $N$  such that  $x = x \triangleleft^N y$  for any  $x, y \in X$ , where we denote by  $\bullet \triangleleft^N y$  the  $N$ -times on the right operation with  $y$ . Note that, if  $X$  is of finite order, it is of type  $t_X$  for some  $t_X \in \mathbb{Z}$ . A map  $f : X \rightarrow Y$  between quandles is a *(quandle) homomorphism*, if  $f(a \triangleleft b) = f(a) \triangleleft f(b)$  for any  $a, b \in X$ . We now refer to some examples of quandles:

**Example 2.1** (Alexander quandle). Any  $\mathbb{Z}[T, T^{-1}]$ -module  $M$  is a quandle with the operation  $x \triangleleft y = y + T(x - y)$  for  $x, y \in M$ , called *Alexander quandle*. This operation  $\bullet \triangleleft y$  means a  $T$ -multiple centered at  $y$ . The type is the minimal  $n$  such that  $T^n = \text{id}_M$ , since  $x \triangleleft^N y = y + T^N(x - y)$ . As a typical example, with a choice of an element  $\omega \in \mathbb{F}_q \setminus \{0, 1\}$ , the quandle of the form  $X = \mathbb{F}_q[T]/(T - \omega)$  is called an *Alexander quandle on a finite field*  $\mathbb{F}_q$ .

**Example 2.2** (Symplectic quandle). Let  $K$  be a field. Let  $\Sigma_g$  be the closed surface of genus  $g$ , and let  $X$  be the first homology with  $K$ -coefficients outside 0, that is,  $X = H^1(\Sigma_g; K) \setminus \{0\} = K^{2g} \setminus \{0\}$ . Denote the standard symplectic form by  $\langle, \rangle : H^1(\Sigma_g; K) \times H^1(\Sigma_g; K) \rightarrow K$ . Then this set  $X$  is made into a quandle by the operation  $x \triangleleft y := \langle x, y \rangle y + x \in X$  for any  $x, y \in X$ , and is called *symplectic quandle (over  $K$ )*. The operation  $\bullet \triangleleft y : X \rightarrow X$  is usually called *transvection* of  $y$ . Note that the quandle  $X$  is of type  $p = \text{Char}(K)$ , since  $x \triangleleft^N y = N \langle x, y \rangle y + x$ . When  $K$  is a finite field  $\mathbb{F}_q$ , we denote the quandle by  $\text{Sp}_q^g$ .

**Example 2.3** (Spherical quandle). Let  $K$  be a field of characteristic  $\neq 2$ . Let  $\langle, \rangle : K^{n+1} \otimes K^{n+1} \rightarrow K$  be the standard symmetric bilinear form. Consider a set of the form

$$S_K^n := \{ x \in K^{n+1} \mid \langle x, x \rangle = 1 \}.$$

We define the operation  $x \triangleleft y$  to be  $2\langle x, y \rangle y - x \in S_K^n$ . This pair  $(S_K^n, \triangleleft)$  is a quandle, and is referred to as *spherical quandle (over  $K$ )*. This operation  $\bullet \triangleleft y$  can be interpreted as a  $180^\circ$ -rotation centered at  $y$ . Hence, the quandle is of type 2. In the finite case  $K = \mathbb{F}_q$ , we denote the quandle by  $S_q^n$ .

As observed above, quandle consists of, figuratively speaking, ‘operations itself centered at  $y \in X$ ’, which can be described as homogenous spaces (see [Joy, §7] for detail).

We next review *the associated group* denoted by  $\text{As}(X)$  [FRS1]. This group is the abstract group defined by generators  $e_x$  labeled by  $x \in X$  modulo the relations  $e_x \cdot e_y = e_y \cdot e_{x \triangleleft y}$  for  $x, y \in X$ . That is, the  $\text{As}(X)$  is presented by

$$\text{As}(X) = \langle e_x \ (x \in X) \mid e_{x \triangleleft y}^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y \ (x, y \in X) \rangle.$$

We fix an action  $\text{As}(X)$  on  $X$  defined by  $x \cdot e_y := x \triangleleft y$  for  $x, y \in X$ . Note the equality

$$e_{x \cdot g} = g^{-1} e_x g \in \text{As}(X) \quad (x \in X, \ g \in \text{As}(X)), \quad (2)$$

by definitions. The orbits of the above action of  $\text{As}(X)$  on  $X$  are called *connected components of  $X$* , denoted by  $\text{O}(X)$ . If the action of  $\text{As}(X)$  on  $X$  is transitive,  $X$  is said to be *connected*. For example, it is known [LN, Proposition 1] that an Alexander quandle  $X$  in Example 2.1 is

connected if and only if  $(1-T)X = X$ . Furthermore it can be easily seen that all the quandles over  $K$  in Examples 2.2, 2.3 are connected.

Note that the group  $\text{As}(X)$  is of infinite order. Actually, there is a split epimorphism

$$\epsilon_X : \text{As}(X) \longrightarrow \mathbb{Z} \quad (3)$$

sending each generators  $e_x$  to  $1 \in \mathbb{Z}$ . Furthermore, if  $X$  is connected, by (2), this  $\epsilon_X$  gives the abelianization  $\text{As}(X)_{\text{ab}} \cong \mathbb{Z}$ . The reader should be keep in mind this epimorphism  $\epsilon_X$ .

## 2.2 Review; Quandle homotopy invariant of links.

We begin reviewing  $X$ -colorings. Let  $X$  be a quandle, and  $D$  an oriented link diagram of a link  $L \subset S^3$ . An  $X$ -coloring of  $D$  is a map  $\mathcal{C} : \{\text{arcs of } D\} \rightarrow X$  satisfying the condition  $\mathcal{C}(\gamma_k) = \mathcal{C}(\gamma_i) \triangleleft \mathcal{C}(\gamma_j)$  at each crossings of  $D$  such as Figure 1. Let  $\text{Col}_X(D)$  denote the set of all  $X$ -colorings of  $D$ . As is well-known, if two diagrams  $D_1, D_2$  are related by Reidemeister moves, we easily construct a canonical bijection  $\text{Col}_X(D_1) \simeq \text{Col}_X(D_2)$ ; see, e.g., [Joy, CJKLS]. The set  $\text{Col}_X(D)$  is well-studied topologically, e.g., it is shown [E1, Lemma 3.14]<sup>2</sup> that, if  $X$  is connected, with a choice of a meridian  $\mathbf{m}_i$  and a longitude  $\mathbf{l}_i \in \pi_1(S^3 \setminus L)$  of each link component ( $1 \leq i \leq \#L$ ), then the set  $\text{Col}_X(D)$  is in 1-1 correspondence with a set

$$\{ f \in \text{Hom}_{\text{gr}}(\pi_1(S^3 \setminus L), \text{As}(X)) \mid f(\mathbf{m}_i) = e_{x_i} \text{ for some } x_i \in X, \ x_i = x_i \cdot f(\mathbf{l}_i) \}. \quad (4)$$



Figure 1: Positive and negative crossings.

Next, we briefly recall the quandle homotopy invariant of links (our formula is a modification the formula in [FRS1]). Let us consider the set,  $\Pi_2(X)$ , of all  $X$ -colorings of all diagrams subject to Reidemeister moves and *the concordance relations* illustrated in Figure 2. Then disjoint unions of  $X$ -colorings make the  $\Pi_2(X)$  into an abelian group, which is closely related to a homotopy group  $\pi_2(BX)$  (see (19)). For any link diagram  $D$ , we have a map  $\Xi_{X,D} : \text{Col}_X(D) \rightarrow \Pi_2(X)$  taking  $\mathcal{C}$  to the class  $[\mathcal{C}]$  in  $\Pi_2(X)$ . If  $X$  is of finite order and  $D$  is a diagram of a link  $L$ , then the *quandle homotopy invariant* of  $L$  is defined as the expression

$$\Xi_X(L) := \sum_{\mathcal{C} \in \text{Col}_X(D)} \Xi_{X,D}(\mathcal{C}) \in \mathbb{Z}[\Pi_2(X)]. \quad (5)$$

Moreover, as is known [RS] (see also [N1, §2]), the homotopy invariant is universal among all the “quandle cocycle invariants with local coefficients” (see [CJKLS, CKS, CEGS, Kab] for these definitions). Hence, to answer what the cocycle invariants are, instead, hereafter we may focus on the study of the homotopy invariant and the abelian group  $\Pi_2(X)$  in detail.

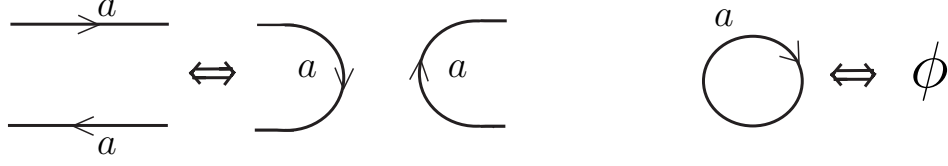


Figure 2: The concordance relations

Finally, we briefly review the original quandle cocycle invariant [CJKLS]. Given a finite quandle  $X$ , we put its quandle homology  $H_2^Q(X)$  with trivial coefficients, which is a quotient of the free module  $\mathbb{Z}\langle X \times X \rangle$  (see §5 for the detailed definition). For an  $X$ -coloring  $\mathcal{C} \in \text{Col}_X(D)$ , we consider a sum  $\sum_{\tau} \epsilon_{\tau}(\mathcal{C}(\gamma_i), \mathcal{C}(\gamma_j)) \in \mathbb{Z}\langle X \times X \rangle$ , where  $\tau$  runs over all crossing of  $D$  as shown in Figure 1 and the symbol  $\epsilon_{\tau} \in \{\pm 1\}$  denotes the sign of the crossing  $\tau$ . As is known (see [RS, N1]), this sum is a 2-cycle, and homology classes of these sums in  $H_2^Q(X)$  are independent of Reidemeister moves and the concordance relations; Hence we obtain a homomorphism

$$\mathcal{H}_X : \Pi_2(X) \longrightarrow H_2^Q(X). \quad (6)$$

Using the formula (5), the quandle cycle invariant of a link  $L$ , denoted by  $\Phi_X(L)$ , is then defined to be the image  $\mathcal{H}_X(\Xi_X(L))$  valued in the group ring  $\mathbb{Z}[H_2^Q(X)]$ . Namely

$$\Phi_X(L) := \mathcal{H}_X(\Xi_X(L)) \in \mathbb{Z}[H_2^Q(X)]. \quad (7)$$

As is known [RS, N1, CKS], given a quandle 2-cocycle  $\phi : X^2 \rightarrow A$ , the pairing between this  $\phi$  and the cycle invariant  $\Phi_X(L)$  coincides with the original cocycle invariant in [CJKLS, Theorem 4.4.].

Although this invariant  $\Phi_X(L)$  is constructed from link diagrams, in §5 we later explain its topological meaning, together with a computation of  $H_2^Q(X)$  following from Eisermann [E1, E2].

### 3 Results on the quandle homotopy invariants

Our purpose in this section is to state our results. In §3.1, we will set up a homomorphism  $\Theta_X$ . In §3.2 we state our results on the group  $\Pi_2(X)$ . As an application, we will see the computations of third quandle homologies in §3.3.

#### 3.1 A key homomorphism $\Theta_X$

Before stating our results, we will set up a homomorphism  $\Theta_X$  in Theorem 3.1, which play a key role in this paper. Furthermore we give Corollary 3.3 which proposes a necessary condition in order to obtain topological interpretations of the quandle homotopy invariants and, hence, of any quandle cocycle invariants.

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<sup>2</sup>In the original paper, the statement is described in knot case. However, we easily see that it holds for links.

To state this theorem, given a closed 3-manifold  $M$  and a group  $G$ , we set up a map from the set of all group homomorphisms,  $\text{Hom}_{\text{gr}}(\pi_1(M), G)$ , to the third group homology  $H_3^{\text{gr}}(G)$  sending  $f$  to  $f_*([M])$ . Here  $[M] \in H_3(M)$  denotes the fundamental class of  $M$ .

**Theorem 3.1.** *Let  $X$  be a connected quandle of type  $t_X$ . Then there is a homomorphism  $\Theta_X : \Pi_2(X) \rightarrow H_3^{\text{gr}}(\text{As}(X))$  satisfying the following property: Any diagram  $D$  of any link  $L \subset S^3$  admits a map  $\theta_{X,D} : \text{Col}_X(D) \rightarrow \text{Hom}_{\text{gr}}(\pi_1(\widehat{C}_L^{t_X}), \text{As}(X))$ , where we denote by  $\widehat{C}_L^{t_X}$  the  $t_X$ -fold cyclic covering space of  $S^3$  branched over the link  $L$ , and provides a commutative diagram, functorial in  $X$ , given by*

$$\begin{array}{ccc} \text{Col}_X(D) & \xrightarrow{\theta_{X,D}} & \text{Hom}_{\text{gr}}(\pi_1(\widehat{C}_L^{t_X}), \text{As}(X)) \\ \Xi_{X,D} \downarrow & & \downarrow (\bullet)_*([\widehat{C}_L^{t_X}]) \\ \Pi_2(X) & \xrightarrow{\Theta_X} & H_3^{\text{gr}}(\text{As}(X)). \end{array}$$

As is seen in the proof, we later construct these maps  $\theta_{X,D}$  and  $\Theta_X$  concretely (see §4).

**Remark 3.2.** As is seen in §4, for any  $X$ -coloring  $\mathcal{C} \in \text{Col}_X(D)$ , the homomorphism  $\theta_{X,D}(\mathcal{C}) : \pi_1(\widehat{C}_L^{t_X}) \rightarrow \text{As}(X)$  factors through the kernel  $\text{Ker}(\epsilon_X) \subset \text{As}(X)$  in (3) and is  $\mathbb{Z}$ -equivariant with respect to the canonical actions  $\mathbb{Z} \curvearrowright \widehat{C}_L^{t_X}$  from the covering transformations and  $\mathbb{Z} \curvearrowright \text{Ker}(\epsilon_X)$  from the splitting (3).

Next, to describe Corollary 3.3, we briefly review the Dijkgraaf-Witten invariant [DW]. Given a finite group  $G$  and a group 3-cocycle  $\kappa \in H_3^{\text{gr}}(G; A)$ , the *Dijkgraaf-Witten invariant* of  $M$  is defined as a formal sum of some pairings expressed as

$$\text{DW}_{\kappa}(M) := \sum_{f \in \text{Hom}(\pi_1(M), G)} \langle \kappa, f_*([M]) \rangle \in \mathbb{Z}[A]. \quad (8)$$

Here  $\mathbb{Z}[A]$  is a group ring of  $A$ . Furthermore, when  $X$  is of finite order, as the image of the  $\theta_{X,D}$  in Theorem 3.1, we define a certain ( $\mathbb{Z}$ -equivariant) part of the Dijkgraaf-Witten invariant of branched covering spaces  $\widehat{C}_L^{t_X}$  as the formula

$$\text{DW}_{\text{As}(X)}^{\mathbb{Z}}(\widehat{C}_L^{t_X}) := \sum_{\mathcal{C} \in \text{Col}_X(D)} [\Theta_X(\Xi_{X,D}(\mathcal{C}))] = \sum_{\mathcal{C} \in \text{Col}_X(D)} \theta_{X,D}(\mathcal{C})_*([\widehat{C}_L^{t_X}]) \in \mathbb{Z}[H_3^{\text{gr}}(\text{As}(X))]. \quad (9)$$

Using this, under an assumption, we discuss the quandle homotopy invariant as follows:

**Corollary 3.3.** *Let  $X$ ,  $\widehat{C}_L^{t_X}$ ,  $\Theta_X$  be as above. Let  $|X| < \infty$ . Take the Hurewicz homomorphism  $\mathcal{H}_X : \Pi_2(X) \rightarrow H_2^Q(X)$  in (6). If the sum  $\Theta_X \oplus \mathcal{H}_X : \Pi_2(X) \rightarrow H_3^{\text{gr}}(\text{As}(X)) \oplus H_2^Q(X)$  is an isomorphism after  $\ell$ -localization for some prime  $\ell$ , then, for any link  $L$ , the  $\ell$ -torsion of the quandle homotopy invariant of  $L$  is decomposed as*

$$(\Theta_X \oplus \mathcal{H}_X)_{(\ell)}(\Xi_X(L)) = \text{DW}_{\text{As}(X)}^{\mathbb{Z}}(\widehat{C}_L^{t_X})_{(\ell)} + \Phi_X(L)_{(\ell)} \in \mathbb{Z}[\Pi_2(X)_{(\ell)}]. \quad (10)$$

To conclude, under the assumption on the sum  $\Theta_X \oplus \mathcal{H}_X$ , we succeed in providing a topological interpretation of the quandle homotopy invariant  $\Xi_X(L)$  as mentioned in the introduction. Actually, the two invariants in the right hand side of (10) are defined topologically.



### 3.2 Results on the homomorphisms $\Theta_X \oplus \mathcal{H}_X$

Following Corollary 3.3, it is thus significant to find quandles such that the  $(\Theta_X \oplus \mathcal{H}_X)_{(\ell)}$  are isomorphisms. This section lists such quandles.

To begin with, we discuss a general argument, and state the main theorem: to be specific,

**Theorem 3.4.** *Let  $X$  be a connected quandle of type  $t_X < \infty$ . If the homology  $H_3^{\text{gr}}(\text{As}(X))$  is finitely generated (e.g., if  $X$  is of finite order), then the homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism modulo  $t_X$ -torsion.*

As a result, combing this with Corollary 3.3, we have obtained the interpretation of some torsion of the quandle homotopy (cocycle) invariant. Precisely,

**Corollary 3.5.** *Let  $X$  be a finite connected quandle of type  $t_X$ . Then the equality (10) in Corollary 3.3 holds for any prime which does not divide  $t_X$ .*

Note that there are many quandles whose type  $t_X$  are powers of some prime, e.g., the quandles in Examples 2.2, 2.3, and connected quandles of order  $\leq 8$ . Hence, for such quandles  $X$ , we determine the most parts of  $\pi_2(BX)$  by Theorem 3.4.

**Remark 3.6.** Corollary 3.5 is a strong generalization of some results in [Kab, HN]. The results dealt with only for the Alexander quandle of the form  $X = \mathbb{Z}[T^{\pm 1}]/(2n - 1, T + 1)$ . Indeed, their arguments were based on peculiar properties of the quandle.

We moreover address some  $t_X$ -torsion subgroups of  $\Pi_2(X)$ . First we discuss an easy condition to vanish these  $t_X$ -torsions:

**Proposition 3.7.** *Let  $X$  be a connected quandle of  $t_X$ . If the  $t_X$ -torsion of the image  $H_3^{\text{gr}}(\text{As}(X)) \oplus H_2^Q(X)$  is assumed to be zero, then that of the map  $\Theta_X \oplus \mathcal{H}_X$  is zero.*

The easy proof will appear in §6.1. However, we give examples satisfying the assumption.

**Example 3.8.** We now discuss regular Alexander quandles  $X$  of finite order. Here,  $X$  is said to be *regular*, if  $X$  is connected and its type is relatively prime to the order  $|X|$ , e.g., the Alexander quandles on  $\mathbb{F}_q$  since  $\omega^{q-1} = 1$ . Then the assumption holds. Actually, the  $H_2^Q(X)$  is a quotient of  $X \otimes_{\mathbb{Z}} X$  (see Proposition B.1), and it can be easily seen that  $t_X$ -torsion of the group homology  $H_*^{\text{gr}}(\text{As}(X))$  is zero by the lower central series of  $\text{As}(X)$  [see (36)]. In particular, by Theorem 3.4, we immediately have

**Corollary 3.9.** *Let  $X$  be a finite regular Alexander quandle. Then the homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism.*

However, in [N2, Appendix] from another direction, the  $\Pi_2(X) \otimes \mathbb{Z}/p$  were computed with respect to the Alexander quandles on  $\mathbb{F}_q$  with  $p \neq 2$ .

We next discuss several quandles  $X$  such that the  $t_X$ -torsions of  $\Pi_2(X)$  are non-zero. Let us deal discuss the symplectic and spherical quandles over  $\mathbb{F}_q$  in Examples 2.2, 2.3. For this, we call  $q = p^d \in \mathbb{N}$  *exceptional*, if the  $q$  is one of  $\{3, 3^2, 3^3, 5, 7\}$ , that is,  $d(p - 1) \leq 6$  (cf. the condition in Theorem 7.3 later).

**Theorem 3.10.** *Let  $q = p^d$  be odd, and be not exceptional.*

- (I) *Let  $X$  be the symplectic quandle  $\mathbf{Sp}_q^n$  over  $\mathbb{F}_q$ . Then the homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism. Furthermore,  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1)$  for  $n > 1$  and  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d$  for  $n = 1$ .*
- (II) *Let  $X$  be the spherical quandle  $S_q^n$  over  $\mathbb{F}_q$ . The homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism modulo 2-torsion. Moreover,  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \bmod 2$  for  $n > 2$ . Furthermore, when  $n = 2$ , we have  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q) \bmod 2$ , where  $\delta_q = \pm 1$  is according to  $q \equiv \pm 1 \pmod{4}$ .*

**Remark 3.11.** We comment the exceptional cases of  $q$ . As is seen in the proofs in §7.1, the homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism modulo  $2p$ -torsion; Furthermore, if  $n$  is enough large, then the  $p$ -torsion part of the  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism (see §7.1 and Remark 7.4).

We furthermore focus on connected quandles  $X$  of order  $\leq 8$  as follows:

**Theorem 3.12.** *For any connected quandle  $X$  of order  $\leq 8$ , the homomorphism  $\Theta_X \oplus \mathcal{H}_X$  is an isomorphism  $\Pi_2(X) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus \text{Im}(\mathcal{H}_X)$ .*

See §7.2 for the proof and for the computations of  $\Pi_2(X)$ . This theorem generalizes some results in the previous paper [N1, §4]. Indeed, the author only estimated the group  $\Pi_2(X)$  for quandles  $X$  with  $|X| \leq 6$ .

In conclusion, for such quandles discussed above, we obtain a topological meaning of the quandle homotopy (cocycle) invariants from the viewpoint of Corollary 3.3.

### 3.3 Application; some computations of third quandle homologies

As an application of computing the (homotopy) group  $\Pi_2(X)$ , we develop a new method to compute third quandle homology  $H_3^Q(X)$  (see §5 for the definition). Moreover, we will give some explicit computation of  $H_3^Q(X)$  for some quandles. The statements in this subsection will be proven in §9.

To describe this, we briefly review *the inner automorphism group*,  $\text{Inn}(X)$ , of a quandle  $X$ . Recalling the action of  $\text{As}(X)$  on  $X$ , we thus have a group homomorphism  $\psi_X$  from  $\text{As}(X)$  to the symmetric group  $\mathfrak{S}_{|X|}$ . The group  $\text{Inn}(X)$  is defined by the image ( $\subset \mathfrak{S}_{|X|}$ ). Hence we have a group extension

$$0 \longrightarrow \text{Ker}(\psi_X) \longrightarrow \text{As}(X) \xrightarrow{\psi_X} \text{Inn}(X) \longrightarrow 0 \quad (\text{exact}). \quad (11)$$

By the equality (2), this kernel  $\text{Ker}(\psi_X)$  is contained in the center. Furthermore, as we later show (Corollary 6.4), if  $X$  is of type  $t_X$  and connected, then  $\text{Ker}(\psi_X) \cong \mathbb{Z} \oplus H_2^{\text{gr}}(\text{Inn}(X))$  modulo  $t_X$ -torsion.

We state a theorem on the third homology  $H_3^Q(X)$  as a rough estimate.

**Theorem 3.13.** *Let  $X$  be a connected quandle of finite order. Then there is the following isomorphism up to  $2|\text{Inn}(X)|/|X|$ -torsion:*

$$H_3^Q(X) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus (\text{Ker}(\psi_X) \wedge \text{Ker}(\psi_X)).$$

Here note that the type  $t_X$  divides the order  $|\text{Inn}(X)|/|X|$  (Lemma A.7). In summary, many torsion subgroups of the third quandle homology are determined after computing the group homologies of  $\text{As}(X)$  and  $\text{Inn}(X)$ .

**Remark 3.14.** The isomorphism does not hold in the 2-torsion subgroup. See Remark 7.6 for a counterexample.

To be more concrete, we will calculate the third quandle homologies of some quandles. Notice that Theorem 3.13 is of use with respect to quandles  $X$  such that the order  $|\text{Inn}(X)|/|X|$  is small. For example, in Alexander case, the order  $|\text{Inn}(X)|/|X|$  equals  $\text{Type}(X)$  exactly (see [N2, Lemma 5.6]). We then describe the  $H_3^Q(X)$  of regular Alexander quandles.

**Theorem 3.15.** *Let  $X$  be a finite regular Alexander quandle. Then there is the isomorphism  $H_3^Q(X) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus (\bigwedge^2 \text{Ker}(\psi_X))$  above modulo 2-torsion.*

*Moreover, if the order of  $X$  is odd, then the 2-torsion subgroups of the both sides are zero.*

Here remark that, the associated group  $\text{As}(X)$  and the kernel  $\text{Ker}(\psi_X)$  have been calculated by Clauwens [Cla2] (see also Appendix B). In particular, the group  $\text{As}(X)$  is a nilpotent group of degree 2 (see (36) for the lower central series); hence its (co)homology is not simple, e.g., it contains some Massey products (see [N4, §4]).

**Example 3.16.** Let  $X$  be the Alexander quandle over  $\mathbb{F}_q$  of odd order, which is regular. Mochizuki [Moc] has computed the third quandle cohomology with  $\mathbb{F}_q$ -coefficients, and gave a polynomial-presentation of its basis. However, his presentation is a little complicated (see polynomials “T” in [Moc, §2.2]). Theorem 3.15 implies that the reason is derived from the third group homology of the nilpotent group  $\text{As}(X)$ .

Next, we further determine explicitly the third quandle homologies of symplectic quandles and spherical quandles over  $\mathbb{F}_q$ , although the orders  $|\text{Inn}(X)|/|X|$  are not simple.

**Theorem 3.17.** *Let  $q = p^d$  be odd and not exceptional.*

(I) *Let  $X = \text{Sp}_q^n$  be the symplectic quandle of order  $q^{2n} - 1$  in Example 2.2. If  $n = 1$ , then*

$$H_2^Q(X) \cong (\mathbb{Z}/p)^d, \quad H_3^Q(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^{d(d+1)/2}.$$

*On the other hand, if  $n > 1$ , then  $H_2^Q(X) \cong H_3^Q(X) \cong 0$ .*

(II) *Let  $X = S_q^n$  be the spherical quandle over  $\mathbb{F}_q$  in Example 2.3. Let  $\delta_q \in \{\pm 1\}$  be according to  $q \equiv \pm 1 \pmod{4}$ . If  $n = 2$ , then*

$$H_2^Q(X) \cong \mathbb{Z}/(q - \delta_q), \quad H_3^Q(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q)$$

*modulo 2-torsion. If  $n > 2$ , then  $H_3^Q(X)$  and  $H_2^Q(X)$  are elementary abelian 2-groups.*

This theorem mostly settles a problem posed by Kabaya [ILD1] for computing the homology  $H_3^Q(\text{Sp}_q^n)$  with  $n = 1$ .

Finally, we study some extended quandles considered in [Joy, §7], which plays a role to prove Theorem 3.4, and compute these quandle homologies. Recall the homomorphism  $\epsilon_X :$

$\text{As}(X) \rightarrow \mathbb{Z}$  in (3). For a connected quandle  $X$  with  $a \in X$ , we equip the kernel  $\text{Ker}(\epsilon_X)$  with a quandle operation by setting

$$g \triangleleft h := e_a^{-1} g h^{-1} e_a h \quad \text{for } g, h \in \text{Ker}(\epsilon_X).$$

We denote the quandle  $(\text{Ker}(\epsilon_X), \triangleleft)$  by  $\tilde{X}$ , which is called *the extended quandle (of  $X$ )*. We see easily the independence of the choice of  $a \in X$  up to quandle isomorphisms. Using the restricted action  $X \curvearrowright \text{Ker}(\epsilon_X) \subset \text{As}(X)$ , the canonical map  $p : \tilde{X} \rightarrow X$  sending  $g$  to  $a \cdot g$  is known to be a quandle homomorphism (see [Joy, Theorem 7.1]), and is called *the universal (quandle) covering of  $X$* , according to Eisermann [E2, §5]. We see that, when  $X$  is finite and of type  $t_X$ , so is the extended one  $\tilde{X}$ . We later show the connectivity of  $\tilde{X}$  (Lemma 6.9).

The theorem on the groups  $H_3(\tilde{X})$  and  $\Pi_2(\tilde{X})$  of extended quandles  $\tilde{X}$  is as follows:

**Theorem 3.18.** *Let  $X$  be a connected quandle of type  $t_X$ . Let  $p : \tilde{X} \rightarrow X$  be the universal covering mentioned above. If  $H_3^{\text{gr}}(\text{As}(X))$  is finitely generated, then there are isomorphisms*

$$H_3^Q(\tilde{X}) \cong \Pi_2(\tilde{X}) \cong H_3^{\text{gr}}(\text{As}(X)) \quad \text{modulo } t_X\text{-torsion.}$$

Here the isomorphism  $\Pi_2(\tilde{X}) \cong H_3^{\text{gr}}(\text{As}(X))$  is obtained from the composite  $\Theta_X \circ p_*$ .

**Remark 3.19.** Furthermore, we will determine the second homology  $H_2^Q(\tilde{X})$  (Theorem 6.12).

In a subsequent paper [N4], this theorem on the quandle  $\tilde{X}$  will be used to understand the key homomorphism  $\Theta_X$  from a viewpoint of complexes of groups and quandles.

## 4 The key homomorphism $\Theta_X$

From now on, we will prove the results mentioned in the previous section.

Our purpose in this section is to prove Theorem 3.1 and, is to construct a homomorphism from  $\Pi_2(X)$  to a bordism group (Lemma 4.2), which plays a key role in this paper. The construction is a modification of a map in [HN, §4.5], where we dealt with only a class of “4-fold symmetric quandles”.

For the purpose, we first describe a presentation of the fundamental group  $\pi_1(\widehat{C}_L^t)$ , where  $\widehat{C}_L^t$  denotes the  $t$ -fold covering of  $S^3$  branched along a link  $L$ . Put a link diagram  $D$  of  $L$ . Let  $\gamma_0, \dots, \gamma_n$  be the arcs of the diagram  $D$ . Let  $\widetilde{S^3 \setminus L}$  be the abelian covering space of  $S^3 \setminus L$  associated with the homomorphism  $\pi_1(S^3 \setminus L) \rightarrow \mathbb{Z}$  sending each  $\gamma_i$  to 1. For an index  $s \in \mathbb{Z}$ , we take a copy  $\gamma_{i,s}$  of the arc  $\gamma_i$ . Then, by Reidemeister-Schreier method (see, e.g., [Rol, Appendix A] and [Kab, §3]), the fundamental group  $\pi_1(\widetilde{S^3 \setminus L})$  can be presented by

$$\text{generators : } \gamma_{i,s} \quad (0 \leq i \leq n, s \in \mathbb{Z}),$$

$$\text{relations : } \gamma_{k,s} = \gamma_{j,s-1}^{-1} \gamma_{i,s-1} \gamma_{j,s} \text{ for each crossings such as Figure 1, and } \gamma_{0,j} = 1.$$

Further we can define an inclusion  $\iota : \pi_1(\widetilde{S^3 \setminus L}) \hookrightarrow \pi_1(S^3 \setminus L)$  by  $\iota(\gamma_{i,s}) = \gamma_0^{s-1} \gamma_i \gamma_0^{-s}$ . Moreover, the fundamental group  $\pi_1(\widehat{C}_L^t)$  is obtained from the presentation of  $\pi_1(\widetilde{S^3 \setminus L})$  by adding the relations  $\gamma_{i,s} = \gamma_{i,s+t}$ .

Next, given a quandle  $X$  of type  $t_X$ , we now construct a map (13) below. For this, given an  $X$ -coloring  $\mathcal{C}$ , let us correspond each arc  $\gamma_i$  to the generator  $\Gamma_{\mathcal{C}}(\gamma_i) := e_{\mathcal{C}(\gamma_i)} \in \text{As}(X)$ . By the Wirtinger presentation, this correspondence defines a group homomorphism

$$\Gamma_{\mathcal{C}} : \pi_1(S^3 \setminus L) \longrightarrow \text{As}(X). \quad (12)$$

Recall the homomorphism  $\epsilon_X : \text{As}(X) \rightarrow \mathbb{Z}$  sending the generators  $e_x$  to  $1 \in \mathbb{Z}$  [see (3)]. Then the restriction of the homomorphism  $\Gamma_{\mathcal{C}}$  on  $\pi_1(\widetilde{S^3 \setminus L})$  factors through the kernel  $\text{Ker}(\epsilon_X)$ . Here we note the following lemma, which is also used later.

**Lemma 4.1.** *Let  $X$  be a connected quandle of type  $t$ . Then, for any  $x, y \in X$ , we have the identity  $(e_x)^t = (e_y)^t$  in the center of  $\text{As}(X)$ .*

*Proof.* For any  $b \in X$ , note the equalities  $(e_x)^{-t} e_b e_x^t = e_{(\dots(b \triangleleft x) \dots) \triangleleft x} = e_b$  in  $\text{As}(X)$ . Namely  $(e_x)^t$  lies in the center. Furthermore, the connectivity admits  $g \in \text{As}(X)$  such that  $x \cdot g = y$ . Hence it follows from (2) that  $(e_x)^t = g^{-1}(e_x)^t g = (e_{x \cdot g})^t = (e_y)^t$  as desired.  $\square$

Thus, by the above presentation of  $\pi_1(\widehat{C}_L^t)$ , the map  $\Gamma_{\mathcal{C}}$  induces a homomorphism  $\widehat{\Gamma}_{\mathcal{C}} : \pi_1(\widehat{C}_L^t) \rightarrow \text{Ker}(\epsilon_X)$ . Precisely, this  $\widehat{\Gamma}_{\mathcal{C}}$  is defined by the formula  $\widehat{\Gamma}_{\mathcal{C}}(\gamma_{i,s}) = e_{\mathcal{C}(\gamma_0)}^{s-1} e_{\mathcal{C}(\gamma_i)} e_{\mathcal{C}(\gamma_0)}^{-s}$ . In summary, we obtain a map

$$\theta_{X,D} : \text{Col}_X(D) \longrightarrow \text{Hom}_{\text{gr}}(\pi_1(\widehat{C}_L^t), \text{Ker}(\epsilon_X)), \quad (\mathcal{C} \longmapsto \widehat{\Gamma}_{\mathcal{C}}). \quad (13)$$

We here remark that this map depends on the choice of  $\gamma_0$ ; However it does not up to conjugacy of  $\text{Ker}(\epsilon_X)$ , if  $X$  is connected.

Finally, in order to state Lemma 4.2 below, we briefly recall the *oriented bordism group*,  $\Omega_n(G)$ , of a group  $G$ . We consider a pair consisting of a closed connected oriented  $n$ -manifold  $M$  without boundary and a homomorphism  $\pi_1(M) \rightarrow G$ . Then a set,  $\Omega_n(G)$ , is defined to be the quotient set of such pairs  $(M, \pi_1(M) \rightarrow G)$  subject to  $G$ -bordant equivalence. Here, such two pairs  $(M_i, f_i : \pi_1(M_i) \rightarrow G)$  are  $G$ -bordant, if there exist an oriented  $(n+1)$ -manifold  $W$  which bounds the connected sum  $M_1 \# (-M_2)$  and a homomorphism  $\bar{f} : \pi_1(W) \rightarrow G$  so that  $f_1 * f_2 = \bar{f} \circ (i_W)_*$ , where  $i_W : M_1 \# (-M_2) \rightarrow W$  is the natural inclusion. An abelian group structure is imposed on  $\Omega_n(G)$  by connected sum. Note that  $\Omega_n(G)$  agrees with the usual oriented ( $SO$ -)bordism group of the Eilenberg-MacLane space  $K(G, 1)$ .

**Lemma 4.2.** *Let  $X$  be a connected quandle of type  $t$ . Then by considering all link diagrams  $D$ , the maps  $\theta_{X,D}$  in (13) give rise to an additive homomorphism*

$$\Theta_{\Pi\Omega} : \Pi_2(X) \longrightarrow \Omega_3(\text{Ker}(\epsilon_X)). \quad (14)$$

*A sketch of the proof.* Since the proof is analogous to [HN, Lemma 5.3 and Proposition 4.3] essentially, we will sketch it. To obtain the homomorphism  $\Theta_{\Pi\Omega}$ , it suffices to show that the maps take the concordance relations to the bordance ones.

First, to deal with the local move in the right of Figure 2, we recall that the  $t$ -fold cyclic covering of  $S^3$  branched over the 2-component trivial link  $T_2$  is  $S^2 \times S^1 \rightarrow S^3$  (see [Rol,

§10.C]). It suffices to show that any  $f : \pi_1(S^2 \times S^1) \rightarrow \text{Ker}(\epsilon_X)$  is  $G$ -bordant. Indeed,  $f : \pi_1(B^3 \times S^1) \rightarrow \text{Ker}(\epsilon_X)$  provides its bordance, where  $B^3$  is a ball.

Next, for two  $X$ -colorings  $\mathcal{C}_1$  and  $\mathcal{C}_2$  related by the left in Figure 3, we will show that the connected sum  $\theta_{X,D}(\mathcal{C}_1 \# (-\mathcal{C}_2)^*) : \pi_1(\widehat{\mathcal{C}}_{L_1}^t \# \widehat{\mathcal{C}}_{L_2}^t) \rightarrow \text{Ker}(\epsilon_X)$  is null-bordance. Let  $N_{\mathcal{C}_i} \subset S^3$  be a neighborhood around the local move. Then we put a canonical saddle  $\mathcal{F}$  in  $N_{\mathcal{C}_1} \times [0, 1]$  which bounds the four arcs illustrated in Figure 3. Define an embedded surface  $W \subset S^3 \times [0, 1]$  by  $((L_1 \setminus N_{\mathcal{C}_1}) \times [0, 1]) \cup \mathcal{F}$ . Then the  $t$ -fold cyclic covering  $\mathcal{W} \rightarrow S^3 \times [0, 1]$  branched over  $W$  bounds  $\widehat{\mathcal{C}}_{L_1}^t \sqcup \widehat{\mathcal{C}}_{L_2}^t$ . Moreover, we can verify that the sum  $\theta_{X,D}(\mathcal{C}_1 \# (-\mathcal{C}_2)^*)$  extends to a group homomorphism  $\pi_1(\mathcal{W}) \rightarrow \text{Ker}(\epsilon_X)$ , which gives the desired null-bordance.  $\square$

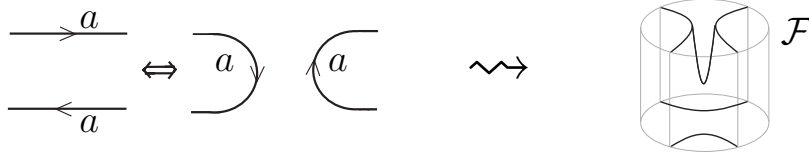


Figure 3:  $\mathcal{F}$  is a saddle in the neighborhood  $N_{\mathcal{C}_i} \times [0, 1]$ .

Finally, in order to prove Theorem 3.1, we recall *Thom homomorphism*  $\mathcal{T}_G : \Omega_n(G) \rightarrow H_n(K(G, 1)) = H_n^{\text{gr}}(G)$  obtained by assigning to every pair  $(M, f : \pi_1(M) \rightarrow G)$  the image of the orientation class under  $f_* : H_n(M) \rightarrow H_n(K(G, 1))$ . It is widely known that, if  $n = 3$ , the homomorphism  $\mathcal{T}_G$  is an isomorphism  $\Omega_3(G) \cong H_3^{\text{gr}}(G)$ .

*Proof of Theorem 3.1.* Let  $G$  be  $\text{As}(X)$ . Put the inclusion  $\iota : \text{Ker}(\epsilon_X) \hookrightarrow \text{As}(X)$ . We define  $\Theta_X$  by the composite  $\mathcal{T}_{\text{As}(X)} \circ \iota_* \circ \Theta_{\Pi\Omega} : \Pi_2(X) \rightarrow H_3^{\text{gr}}(\text{As}(X))$ . From the construction from the maps  $\theta_{X,D}$ , the required commutative diagram holds.  $\square$

## 5 Preliminaries; quandle homology and cocycle invariant.

As a preparation, we now review some properties of the (co)homology of the rack space, and a topological interpretation of the cocycle invariants. There is nothing new in this section.

A set  $Y$  acted on by  $\text{As}(X)$  is called  $X$ -set. For example, the quandle  $X$  is itself an  $X$ -set, referred as to *the primitive  $X$ -set*, from the canonical action  $X \curvearrowright \text{As}(X)$  mentioned in §2.

For this, we begin reviewing the (action) rack space  $B(X, Y)$  introduced by Fenn-Rourke-Sanderson [FRS1, Example 3.1.1]. Fix a quandle  $X$  and an  $X$ -set  $Y$  equipped with their discrete topology. We put a union  $\bigcup_{n \geq 0} (Y \times ([0, 1] \times X)^n)$ , and consider the relations given by

$$(y, t_1, x_1, \dots, x_{j-1}, 1, x_j, t_{j+1}, \dots, t_n, x_n) \sim (y \cdot e_{x_j}, t_1, x_1 \triangleleft x_j, \dots, t_{j-1}, x_{j-1} \triangleleft x_j, t_{j+1}, x_{j+1}, \dots, t_n, x_n),$$

$$(y, t_1, x_1, \dots, x_{j-1}, 0, x_j, t_{j+1}, \dots, t_n, x_n) \sim (y, t_1, x_1, \dots, t_{j-1}, x_{j-1}, t_{j-1}, x_{j+1}, \dots, t_n, x_n).$$

Then the *rack space*  $B(X, Y)$  is defined to be the quotient space. When  $Y$  is a single point, we denote the space by  $BX$  for short. By construction, we have a cell decomposition of  $B(X, Y)$ ,

regarding the projection  $\bigcup_{n \geq 0} (Y \times ([0, 1] \times X^n) \rightarrow B(X, Y)$  as characteristic maps (see [N1, §2] or [N2, §2.2] for a detailed picture of the 3-skeleton of  $BX$ ).

Furthermore, we briefly review the rack and quandle (co)homologies (our formula relies on [AG, CEGS]). Let  $X$  be a quandle, and  $Y$  an  $X$ -set. Let  $C_n^R(X, Y)$  be the free right  $\mathbb{Z}$ -module generated by  $Y \times X^n$ . Define a boundary  $\partial_n^R : C_n^R(X, Y) \rightarrow C_{n-1}^R(X, Y)$  by

$$\partial_n^R(y, x_1, \dots, x_n) = \sum_{1 \leq i \leq n} (-1)^i ((y \triangleleft x_i, x_1 \triangleleft x_i, \dots, x_{i-1} \triangleleft x_i, x_{i+1}, \dots, x_n) - (y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)).$$

The composite  $\partial_{n-1}^R \circ \partial_n^R$  is known to be zero. The homology is denoted by  $H_n^R(X, Y)$  and is called *rack homology*. As is known, the cellular complex of the rack space  $B(X, Y)$  above is isomorphic to the rack complex  $(C_*^R(X, Y), \partial_*^R)$ .

**Remark 5.1.** If  $Y$  is the primitive  $X$ -set  $Y = X$ , we have the chain isomorphism  $C_n^R(X, X) \rightarrow C_{n+1}^R(X, \text{pt})$  induced from the identification  $X \times X^n \simeq X^{n+1}$ . In particular, we obtain an isomorphism  $H_n^R(X, X) \cong H_{n+1}^R(X, \text{pt})$  (see, e.g., [Cla1, Proposition 2.1]).

Furthermore, let  $C_n^D(X, Y)$  be a submodule of  $C_n^R(X, Y)$  generated by  $(n+1)$ -tuples  $(y, x_1, \dots, x_n)$  with  $x_i = x_{i+1}$  for some  $i \in \{1, \dots, n-1\}$ . Then it can be easily seen that the  $C_n^D(X, Y)$  is a subcomplex of  $C_n^R(X, Y)$ . The *quandle homology*,  $H_n^Q(X, Y)$ , is defined by the homology of the quotient complex  $C_n^R(X, Y)/C_n^D(X, Y)$ .

We will review some properties of these homologies in the case where  $Y$  is a single point (In what follows, we suppress the symbol  $Y$ ). Let us decompose  $X$  as  $X = \sqcup_{i \in O(X)} X_i$  by the connected components. The following direct sum decompositions were shown [LN]:

$$H_1^R(X) \cong \mathbb{Z}^{O(X)}, \quad H_2^R(X) \cong H_2^Q(X) \oplus \mathbb{Z}^{O(X)}. \quad (15)$$

Furthermore, Eisermann [E2] gave a computation of the second quandle homologies  $H_2^Q(X)$  with trivial  $\mathbb{Z}$ -coefficients. To see this, for  $i \in O(X)$ , define a homomorphism

$$\epsilon_i : \text{As}(X) \rightarrow \mathbb{Z} \quad \text{by} \quad \begin{cases} \epsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\ \epsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i. \end{cases} \quad (16)$$

Note that the sum  $\bigoplus_{i \in O(X)} \epsilon_i$  yields the abelianization  $\text{As}(X)_{\text{ab}} \cong \mathbb{Z}^{O(X)}$  by (2). Furthermore

**Theorem 5.2** ([E2, Theorem 9.9]). *Let  $X$  be a quandle. Decompose  $X = \sqcup_{i \in O(X)} X_i$  as the orbits by the action of  $\text{As}(X)$ . Fix an element  $x_i \in X_i$  for each  $i \in O(X)$ . Let  $\text{Stab}(x_i) \subset \text{As}(X)$  be the stabilizer of  $x_i$ . Then the quandle homology  $H_2^Q(X)$  is isomorphic to the direct sum of the abelianizations of  $\text{Stab}(x_i) \cap \text{Ker}(\epsilon_i)$ : Namely,  $\bigoplus_{i \in O(X)} (\text{Stab}(x_i) \cap \text{Ker}(\epsilon_i))_{\text{ab}}$ .*

Eisermann showed topologically the result using a certain CW-complex. In §B.1 we later give another proof as a slight application of Proposition 9.2.

We furthermore change to the study of the cycle invariant  $\Phi_X(L)$  of links using  $H_2^Q(X)$  explained in (7). We now briefly explain a topological interpretation of this invariant shown by Eisermann [E1, E2]. Decompose  $X = \sqcup_{i \in O(X)} X_i$  as above. Given an  $X$ -coloring  $\mathcal{C} \in \text{Col}_X(D)$ , with respect to a link component of  $L$ , we fix an arc  $\gamma_j$  on  $D$  for  $1 \leq j \leq \#L$ . Let  $x_j := \mathcal{C}(\gamma_j) \in X_j$ , and fix a longitude  $\mathfrak{l}_j$  of the component. Recall from (12) the associated

group homomorphism  $\Gamma_{\mathcal{C}} : \pi_1(S^3 \setminus L) \rightarrow \text{As}(X)$ . Remark that each longitude  $\mathfrak{l}_j$  commutes with the meridian in the same link component. Accordingly,  $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$  commutes with  $e_{x_j}$  in  $\text{As}(X)$ : in other words,  $\Gamma_{\mathcal{C}}(\mathfrak{l}_j) \in \text{Stab}(x_j)$ . Furthermore, since the class of the longitude  $\mathfrak{l}_j$  in  $H_1(S^3 \setminus L)$  is zero, the  $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$  is contained in the kernel  $\text{Ker}(\epsilon_j)$  [see (16)]. Therefore the  $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$  lies in  $\text{Stab}(x_j) \cap \text{Ker}(\epsilon_j)$ . Further consider the class  $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$  in the abelianization of this  $\text{Stab}(x_j) \cap \text{Ker}(\epsilon_j)$ . In summary, we obtain

$$([\Gamma_{\mathcal{C}}(\mathfrak{l}_1)], \dots, [\Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})]) \in \bigoplus_{1 \leq j \leq \#L} (\text{Stab}(x_j) \cap \text{Ker}(\epsilon_j))_{\text{ab}}. \quad (17)$$

Here note that each direct summand in the right side is contained in  $H_2^Q(X)$  by Theorem 5.2. We then put the product  $[\Gamma_{\mathcal{C}}(\mathfrak{l}_1) \cdots \Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})] \in H_2^Q(X)$ . By the discussion in [E1, Theorems 3.24 and 3.25], it can be seen that the product coincides with the value  $\mathcal{H}_X(\mathcal{C})$  in (6) exactly. Hence, when  $|X| < \infty$ , the cycle invariant  $\Phi_X(L)$  written in (7) is reformulated as

$$\Phi_X(L) = \sum_{\mathcal{C} \in \text{Col}_X(D)} [\Gamma_{\mathcal{C}}(\mathfrak{l}_1) \cdots \Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})] \in \mathbb{Z}[H_2^Q(X)]. \quad (18)$$

This was called ‘‘colouring polynomials’’ in [E1, §1]. As a result, this formula suggests an easy computation and a topological meaning of the cycle invariant as desired.

Finally, we observe a relation between this formula and the Hurewicz homomorphism of  $BX$ . Recall the map  $\mathcal{H}_X : \Pi_2(X) \rightarrow H_2^Q(X)$  in (6). Using the isomorphisms (15) and (19), we put a composite

$$\pi_2(BX) \cong \Pi_2(X) \oplus \mathbb{Z}^{O(X)} \xrightarrow{\text{proj.}} \Pi_2(X) \xrightarrow{\mathcal{H}_X} H_2^Q(X) \hookrightarrow H_2^Q(X) \oplus \mathbb{Z}^{O(X)} \cong H_2(BX).$$

From the definitions of  $\mathcal{H}_X$  and the 2-skeleton of the rack space  $BX$ , we easily verify that this composite coincides with the Hurewicz map of  $BX$  modulo the direct summand  $\mathbb{Z}^{O(X)}$  (see [RS] and [N1, Proposition 3.12] for details). In conclusion, the formula (18) enables us to compute the Hurewicz map of  $BX$ .

## 6 Proof of Theorem 3.4

This section proves Theorem 3.4. Since the proof is ad hoc, the hasty reader may read only the outline in §6.1 and skip the details in other subsections. In §6.2 we state an  $t_X$ -vanishing theorem following the outline. In §6.3 we observe some properties of quandle coverings, since they play a key role in the proof. In §6.4, we will investigate the homomorphism  $\Theta_X$  in terms of relative bordism groups, and complete the proof.

### 6.1 Outline of proofs of Theorem 3.4

We roughly outline the proof of Theorem 3.4 to compute  $\Pi_2(X)$ .

As an approach to the homotopy group  $\pi_2(BX)$ , the reader should keep in mind the following isomorphism shown by [FRS2] (see also [N1, Theorem 6.2] for the detailed description):

$$\pi_2(BX) \cong \Pi_2(X) \oplus \mathbb{Z}^{O(X)}, \quad (19)$$



where the symbol  $O(X)$  is the cardinality of the orbits  $X \curvearrowright \text{As}(X)$ . According to the isomorphism (19), to compute  $\Pi_2(X)$ , we will change a focus on computing the homotopy group  $\pi_2(BX)$  from a routine discussion of “Postnikov tower on  $BX$ ”. To illustrate, let  $c : BX \hookrightarrow K(\pi_1(BX), 1)$  be an inclusion obtained by killing the higher homotopy groups of  $BX$ . Notice that the homotopy fiber is the universal covering of  $BX$ . Thanks to the fact [FRS2] that the action of  $\pi_1(BX)$  on  $\pi_*(BX)$  is trivial (see also [Cla1, Proposition 2.16]), the Leray-Serre spectral sequence of the map  $c$  gives rise to an exact sequence

$$H_3(BX) \xrightarrow{c_*} H_3^{\text{gr}}(\pi_1(BX)) \longrightarrow \pi_2(BX) \xrightarrow{\mathcal{H}} H_2(BX) \xrightarrow{c_*} H_2^{\text{gr}}(\pi_1(BX)) \rightarrow 0 \quad (\text{exact}), \quad (20)$$

where  $\mathcal{H}$  is the Hurewicz map of  $BX$  (see, e.g., [McC, §8.3<sup>bis</sup>], [Bro, §II.5] for details).

We now reduce this (20) to (21) below. Recall an isomorphism  $H_2(BX) \cong \mathbb{Z}^{O(X)} \oplus H_2^Q(X)$  (see (15)), and the Hurewicz map  $\mathcal{H}$  restricted on the summand  $\mathbb{Z}^{O(X)} \subset \pi_2(BX)$  is shown to be an isomorphism [N1, Proposition 3.12]. Therefore, recalling the isomorphism  $\text{As}(X) \cong \pi_1(BX)$ , the sequence (20) is reduced to be

$$H_3(BX) \xrightarrow{c_*} H_3^{\text{gr}}(\text{As}(X)) \longrightarrow \Pi_2(X) \xrightarrow{\mathcal{H}} H_2^Q(X) \xrightarrow{c_*} H_2^{\text{gr}}(\text{As}(X)) \rightarrow 0 \quad (\text{exact}). \quad (21)$$

Since this paper often uses this sequence, we call it *P-sequence* (of  $X$ ).

Using the *P-sequence*, we outline the proof of Theorem 3.4. Let  $X$  be connected and of type  $t_X < \infty$ . We later show Theorem 6.1 which says that the maps  $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\pi_1(BX))$  in (20) are annihilated by  $t_X$  for  $n \leq 3$ . Thus, the *P-sequence* (21) becomes a short exact sequence modulo  $t_X$  (Corollary 6.3). Hence, in order to obtain  $\Pi_2(X) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus H_2^Q(X)$  without  $t_X$ -torsion as stated in Theorem 3.4, it suffices to show that the homomorphism  $\Theta_X : \Pi_2(X) \rightarrow H_3^{\text{gr}}(\text{As}(X))$  constructed in Theorem 3.1 turns out to be a splitting of the exact sequence (21).

To this end, we first show the splitting with respect to the extended quandles  $\tilde{X}$  (Proposition 6.13). So we will study properties of the  $\tilde{X}$  in §6.3. The point is that, using these properties, the delta map  $\delta_*$  in (21) can be regarded as an inverse of  $\Theta_X$  in a (relative) bordism theory. After that, for general connected quandles, the functoriality of the projection  $\tilde{X} \rightarrow X$  readily completes the proof of Theorem 3.4.

Before going to the next subsection, we briefly prove Proposition 3.7.

*Proof of Proposition 3.7.* Since the  $t_X$ -torsion of  $H_3^{\text{gr}}(\text{As}(X)) \oplus H_2^Q(X)$  is zero by assumption, the  $t_X$ -torsion of  $\Pi_2(X)$  vanishes by (21). Hence, that of the map  $\Theta_X \oplus \mathcal{H}_X$  is zero.  $\square$

## 6.2 The vanishing of the $t_X$ -multiple of the map $c_*$

Following the outline, we now fix some notation in this section and state Theorem 6.1:

**Notation** Let  $X$  be a connected quandle of type  $t_X < \infty$  (possibly,  $X$  is of infinite order). Furthermore,  $\ell$  means a prime which is relatively prime to the  $t_X$ .

**Theorem 6.1.** *Let  $X$  be a connected quandle of type  $t_X < \infty$ . For  $n = 2$  and 3, the induced map  $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$  in (20) is annihilated by  $t_X$ .*

**Remark 6.2.** This theorem is more powerful than a result of Clauwens [Cla1, Proposition 4.4], which stated that, if a finite quandle  $X$  satisfies a certain strong condition, then the composite  $(\psi_X)_* \circ c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X)) \rightarrow H_n^{\text{gr}}(\text{Inn}(X))$  is annihilated by  $|\text{Inn}(X)|/|X|$  for any  $n \in \mathbb{Z}_{\geq 0}$ . Here note that  $t_X$  is a divisor of the order  $|\text{Inn}(X)|/|X|$  (Lemma A.7).

The proof will appear in Appendix C. Instead, we now give some corollaries.

**Corollary 6.3.** *Let  $X$  and  $\ell \in \mathbb{Z}$  be as above. Then the  $P$ -sequence localized at  $\ell$  is reduced to be a short exact sequence*

$$0 \longrightarrow H_3^{\text{gr}}(\text{As}(X))_{(\ell)} \longrightarrow \pi_2(BX)_{(\ell)} \xrightarrow{\mathcal{H}_X} H_2(BX)_{(\ell)} \longrightarrow 0.$$

**Corollary 6.4.** *For any connected quandle  $X$  of type  $t_X$ , the second group homology  $H_2^{\text{gr}}(\text{As}(X))$  is annihilated by  $t_X$ . In particular, the abelian kernel  $\text{Ker}(\psi_X)$  in (11) is isomorphic to  $\mathbb{Z} \oplus H_2^{\text{gr}}(\text{Inn}(X))$  modulo  $t_X$ -torsion.*

*Proof.* By (20), the map  $c_* : H_2(BX) \rightarrow H_2^{\text{gr}}(\text{As}(X))$  is surjective; hence, it follows from Theorem 6.1 that  $H_2^{\text{gr}}(\text{As}(X))$  is annihilated by  $t_X$ . Furthermore, the isomorphism  $\text{Ker}(\psi_X) \cong \mathbb{Z} \oplus H_2^{\text{gr}}(\text{Inn}(X)) \pmod{t_X}$  is immediately obtained from the inflation-restriction exact sequence with respect to the  $\psi_X$  and  $H_1^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}$ .  $\square$

**Corollary 6.5.** *Let  $X$  and  $\ell \in \mathbb{Z}$  be as above. Let  $X$  be of finite order. Then the quandle cycle invariant  $\Phi_X$  in (7) is non-trivial in the  $\ell$ -torsion part. That is, for any class  $[O] \in H_2(BX)_{(\ell)}$ , there exists some  $X$ -coloring  $\mathcal{C}$  of a link such that  $\mathcal{H}_X([\mathcal{C}]) = [O]$ .*

*Proof.* By Corollary 6.3, the map  $\mathcal{H}_X$  localized at  $\ell$  is surjective. Since the  $\Pi_2(X)$  is generated by  $X$ -colorings of links by definition, we have  $\mathcal{H}_X([\mathcal{C}]) = [O]$  for some  $X$ -coloring  $\mathcal{C}$ .  $\square$

**Remark 6.6.** By this discussion, we see that, for a quandle  $X$  with  $H_2^{\text{gr}}(\text{As}(X)) = 0$ , any class  $[O] \in H_2(BX)$  ensures some  $X$ -coloring  $\mathcal{C}$  such that  $\mathcal{H}_X([\mathcal{C}]) = [O]$ .

### 6.3 Some properties of quandle coverings

As preliminaries, we will explore some properties of quandle coverings. Here, recall that a quandle epimorphism  $p : Y \rightarrow Z$  is a (*quandle*) *covering* in the sense of [E2], if the equality  $p(\tilde{x}) = p(\tilde{y}) \in Z$  implies  $\tilde{a} \triangleleft \tilde{x} = \tilde{a} \triangleleft \tilde{y} \in Y$  for any  $\tilde{a}, \tilde{x}, \tilde{y} \in Y$ . For example,

**Example 6.7.** The universal quandle covering  $\tilde{X} \rightarrow X$  explained in §2.2 is a covering.

**Proposition 6.8.** *For any quandle covering  $p : Y \rightarrow Z$ , the induced group surjection  $p_* : \text{As}(Y) \rightarrow \text{As}(Z)$  is a central extension. Furthermore, if  $Y$  and  $Z$  are connected and  $Z$  is of type  $t_X$ , then the abelian kernel  $\text{Ker}(p_*)$  is annihilated by  $t_X$ .*

*Proof.* For any  $y \in Z$ , put arbitrary  $y_i, y_j \in p^{-1}(y)$ . Since  $p$  is a covering, we notice

$$e_{y_i}^{-1} e_b e_{y_i} = e_{b \triangleleft y_i} = e_{b \triangleleft y_j} = e_{y_j}^{-1} e_b e_{y_j} \in \text{As}(Y)$$

for any  $b \in Y$ . Namely, for any indices  $i, j$ , there are central elements  $z_{ij} \in \text{As}(Y)$  such that  $e_{y_i} = z_{ij}e_{y_j}$ . Hence  $\text{As}(Y)$  is generated by  $e_{y_i}$  for  $y \in Z$  and the central elements  $z_{ij}$  with  $i \neq j$ ; Consequently the surjection  $p_*$  is a central extension.

We will show the latter part. Take the inflation-restriction exact sequence, i.e.,

$$\rightarrow H_2^{\text{gr}}(\text{As}(Z)) \xrightarrow{\delta_*} \text{Ker}(p_*) \rightarrow H_1^{\text{gr}}(\text{As}(Y)) \xrightarrow{p_{*,1}^{\text{gr}}} H_1^{\text{gr}}(\text{As}(Z)) \rightarrow 0 \quad (\text{exact}).$$

By connectivities the map  $p_{*,1}^{\text{gr}}$  from  $H_1^{\text{gr}}(\text{As}(Y)) = \mathbb{Z}$  is an isomorphism. Further, since Corollary 6.4 says that  $H_2^{\text{gr}}(\text{As}(Z))$  is annihilated by  $t_X$ , so is the kernel  $\text{Ker}(p_*)$  as desired.  $\square$

From now on, we will focus on studying the universal covering  $p : \tilde{X} \rightarrow X$  in Example 6.7. For this, we first construct an action of  $\text{As}(X)$  on  $\tilde{X}$ . Define a map  $\mu : \text{As}(X) \rightarrow \text{Map}(\tilde{X}, \tilde{X})$  by  $\mu(e_x)(\tilde{y}) := e_a^{-1}\tilde{y}e_x$ , for  $x \in X$  and  $\tilde{y} \in \tilde{X}$ . By definition, the image of  $\mu$  is contained in  $\text{Inn}(\tilde{X})$ , and the composite  $\mu \circ p$  coincides with the canonical action  $\tilde{X} \curvearrowright \text{As}(\tilde{X})$ .

**Lemma 6.9.** *For any connected quandle  $X$ , the action  $\mu$  of  $\text{As}(X)$  on  $\tilde{X}$  is transitive. In particular, the quandle  $\tilde{X}$  is also connected.*

*Proof.* We will show that the identity  $1_{\tilde{X}} \in \tilde{X} = \text{Ker}(\epsilon_X)$  is transitive to any element  $h$  in the quandle  $\tilde{X}$ . Expand  $h \in \tilde{X} \subset \text{As}(X)$  as  $h = e_{x_1}^{\sigma_1} \cdots e_{x_n}^{\sigma_n}$  for some  $x_i \in X$  and  $\sigma_i \in \{\pm 1\}$  with  $i \leq n$ . Since  $h \in \text{Ker}(\epsilon_X)$ , note  $\sum \sigma_i = 0$ . The connectivity ensures some  $g_i \in \text{As}(X)$  so that  $a \cdot g_i^{\sigma_i} = x_i$ . Therefore  $g_i^{-\sigma_i}e_a g_i^{\sigma_i} = e_{a \cdot g_i^{\sigma_i}} = e_{x_i}^{\sigma_i}$  (see (2)). In the sequel, we have

$$1_{\tilde{X}} \cdot (g_1^{\sigma_1} \cdots g_n^{\sigma_n}) = e_a^{\sum \sigma_i} 1_{\tilde{X}} (g_1^{-\sigma_1} e_a g_1^{\sigma_1}) \cdots (g_n^{-\sigma_n} e_a g_n^{\sigma_n}) = e_{x_1}^{\sigma_1} \cdots e_{x_n}^{\sigma_n} = h \in \tilde{X}.$$

This equalities mean the transitivity.  $\square$

Hence we have

**Proposition 6.10.** *For any connected quandle  $X$  of type  $t_X$ , the universal covering  $p : \tilde{X} \rightarrow X$  induces an isomorphism  $p_* : H_3^{\text{gr}}(\text{As}(\tilde{X})) \cong H_3^{\text{gr}}(\text{As}(X)) \bmod t_X$ .*

*Proof.* By connectivity of  $\tilde{X}$  in Lemma 6.9,  $H_2^{\text{gr}}(\text{As}(\tilde{X}))$  and  $H_2^{\text{gr}}(\text{As}(X))$  are annihilated by  $t_X$ . Furthermore, since the  $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$  is a central extension whose kernel is annihilated by  $t_X$  (Proposition 6.8). Therefore we easily have the isomorphism  $p_* : H_3^{\text{gr}}(\text{As}(\tilde{X})) \cong H_3^{\text{gr}}(\text{As}(X)) \bmod t_X$  by the Lyndon-Hochschild sequence of  $p$ .  $\square$

Finally we will determine the second quandle homology of  $\tilde{X}$  (Theorem 6.12). For this,

**Proposition 6.11.** *Let  $X$  be a connected quandle, and  $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$  be the epimorphism induced from the covering  $p : \tilde{X} \rightarrow X$ . Then, under the canonical action of  $\text{As}(\tilde{X})$  on  $\tilde{X}$ , the stabilizer  $\text{Stab}(1_{\tilde{X}})$  of  $1_{\tilde{X}}$  is equal to  $\mathbb{Z} \times \text{Ker}(p_*)$  in  $\text{As}(\tilde{X})$ . Furthermore the summand  $\mathbb{Z}$  is generated by  $1_{\tilde{X}}$ .*

*Proof.* We easily see that the stabilizer of  $1_{\tilde{X}}$  via the action  $\text{Ker}(\epsilon_X) = \tilde{X} \curvearrowright \text{As}(X)$  above is  $\underline{\text{Stab}}(1_{\tilde{X}}) = \{e_a^n\}_{n \in \mathbb{Z}} \subset \text{As}(X)$  exactly. Notice that any central extension of  $\mathbb{Z}$  is trivial; Since the  $p_*$  is a central extension (Proposition 6.8), the restriction  $p_* : \text{Stab}(1_{\tilde{X}}) \rightarrow \underline{\text{Stab}}(1_{\tilde{X}}) = \mathbb{Z}$  means the required identity  $\text{Stab}(1_{\tilde{X}}) = \mathbb{Z} \times \text{Ker}(p_*)$ .  $\square$

**Theorem 6.12.** *Let  $X$  be a connected quandle. Then the second quandle homology of the extended quandle  $\tilde{X}$  is isomorphic to the kernel of the induced map  $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$ . Namely  $H_2^Q(\tilde{X}) \cong \text{Ker}(p_*)$ . In particular,  $H_2^Q(\tilde{X})$  is annihilated by its type  $t_X$ , according to Proposition 6.8.*

*Proof.* Note that  $\tilde{X}$  is connected (Lemma 6.9) and the kernel  $\text{Ker}(p_*)$  is abelian (Proposition 6.8). Accordingly, the desired isomorphisms  $H_2^Q(\tilde{X}) \cong (\text{Ker}(\epsilon_{\tilde{X}}) \cap \text{Stab}(1_{\tilde{X}}))_{\text{ab}} = \text{Ker}(p_*)$  follows immediately from Proposition 6.11 and Theorem 5.2.  $\square$

#### 6.4 The homomorphism $\Theta_X$ as a splitting

We first prove Theorem 3.4 using the following proposition.

**Proposition 6.13.** *Let  $X$  be a connected quandle of type  $t_X < \infty$ , and let  $p : \tilde{X} \rightarrow X$  be the universal covering of  $X$ . If the homology  $H_3^{\text{gr}}(\text{As}(X))$  is finitely generated, then the homomorphism  $\Theta_{\tilde{X}} : \Pi_2(\tilde{X}) \rightarrow \Omega_3(\text{As}(\tilde{X}))$  constructed in Theorem 3.1 is an isomorphism modulo  $t_X$ -torsion and is a splitting in the short exact sequence in Corollary 6.3.*

*Proof of Theorem 3.4.* We put the  $P$ -sequences with respect to  $p : \tilde{X} \rightarrow X$ :

$$\begin{array}{ccccccc} H_3^{\text{gr}}(\text{As}(\tilde{X}))_{(\ell)} & \xrightarrow{\tilde{\tau}_*} & \Pi_2(\tilde{X})_{(\ell)} & \longrightarrow & H_2^Q(\tilde{X})_{(\ell)} & \longrightarrow & H_2^{\text{gr}}(\text{As}(\tilde{X}))_{(\ell)} = 0 \\ p_* \downarrow & & p_* \downarrow & & p_* \downarrow & & p_* \downarrow \\ H_3^{\text{gr}}(\text{As}(X))_{(\ell)} & \xrightarrow{\tau_*} & \Pi_2(X)_{(\ell)} & \longrightarrow & H_2^Q(X)_{(\ell)} & \longrightarrow & H_2^{\text{gr}}(\text{As}(X))_{(\ell)} = 0. \end{array}$$

Since the left  $p_*$  between group homologies is an isomorphism modulo  $t_X$  by Proposition 6.10, the sum  $\Theta_X \oplus \mathcal{H}_X : \Pi_2(X)_{(\ell)} \rightarrow H_3^{\text{gr}}(\text{As}(X))_{(\ell)} \oplus H_2^Q(X)_{(\ell)}$  is an isomorphism by the functoriality of  $\Theta_X$  and Proposition 6.13.  $\square$

So we aim to prove Proposition 6.13. For this, we will review a classical bordism theory (see [CF, §1.4]). For a space-pair  $(Y, A)$  with  $A \subset Y$ , consider a continuous map

$$f : (M, \partial M) \rightarrow (Y, A),$$

where  $M$  is an oriented compact  $n$ -manifold. Such two maps  $f_1, f_2$  are  $G$ -bordant, if there exist an oriented compact manifold  $W$  of dimension  $n+1$  and a map  $F : W \rightarrow Y$  for which

- (I) There is an  $n$ -dimensional submanifold  $M' \subset \partial W$  satisfying  $F(\overline{\partial W} \setminus M') \subset A$ .
- (II) There is a diffeomorphism  $g : (-M_1) \cup M_2 \rightarrow M'$  preserving orientation such that  $(-f_1) \cup f_2 = (F|_{M'}) \circ g$ .

Then the *bordism group* of  $(Y, A)$ , denoted by  $\Omega_n(Y, A)$ , is defined to be the set of all such map  $f$  modulo the  $G$ -bordant relations. An abelian group is imposed on  $\Omega_n(Y, A)$  by disjoint union. Furthermore,  $\{\Omega_n(Y, A)\}_{n \geq 0}$  gives a homology theory (see [CF, §1.6]), and the isomorphism  $\Omega_n(Y, A) \cong H_n(Y, A)$ , for  $n \leq 3$ , is obtained by the Atiyah-Hirzebruch spectral sequence. Furthermore, if  $Y$  is the Eilenberg-MacLane space  $K(G, 1)$  and  $A$  is the empty set, then  $\Omega_n(Y, A)$  is isomorphic to the group  $\Omega_n(G)$  introduced in §4.

Furthermore, we will construct a homomorphism in (22) and describe Lemma 6.14 below. Given an  $\tilde{X}$ -coloring  $\mathcal{C}$  with  $\#L$  link components, we take  $t_X$ -copies of  $\mathcal{C}$ , and denote them by  $\mathcal{C}_j$  for  $1 \leq j \leq m$ . For each link component of  $\mathcal{C}$ , we fix an arc, and consider a sum of  $\mathcal{C}_1, \dots, \mathcal{C}_m$  connected by the arcs (see Figure 4). Denote the resulting link by  $\overline{L}$  and the  $\tilde{X}$ -coloring of  $\overline{L}$  by  $\overline{\mathcal{C}}$ . We then set a homomorphism  $\Gamma_{\overline{\mathcal{C}}} : \pi_1(S^3 \setminus \overline{L}) \rightarrow \text{As}(\tilde{X})$  discussed in (12). Each meridian of  $\overline{\mathcal{C}}$  is sent to be  $e_a$  for some  $a \in X$  by definition. Furthermore, we claim that each longitudes  $\mathfrak{l}_j \in \pi_1(S^3 \setminus \overline{L})$  are sent to be zero. Actually, the formula (17) says that the  $\Gamma_{\overline{\mathcal{C}}}(\mathfrak{l}_j)$  lies in  $\text{Ker}(\epsilon_{\tilde{X}}) \cap \text{Stab}(1_{\tilde{X}})$ , which is equal to the abelian kernel  $\text{Ker}(p_*)$  and is annihilated by  $t_X$  (Propositions 6.8 and 6.11). Consequently, the map  $\Gamma_{\overline{\mathcal{C}}}$  sends every boundaries of  $S^3 \setminus \overline{L}$  to the 1-skeleton of  $B\tilde{X}$ . Here note that the 1-skeleton  $B\tilde{X}_1$  is, by definition, a bouquet of circles labeled by elements of  $\tilde{X}$ . In the sequel, considering all  $\tilde{X}$ -coloring  $\overline{\mathcal{C}}$  and such homomorphisms  $\Gamma_{\overline{\mathcal{C}}}$  modulo the bordance relations, the map  $\mathcal{C} \mapsto \Gamma_{\overline{\mathcal{C}}}$  defines the desired homomorphism

$$\Upsilon_{\tilde{X}} : \Pi_2(\tilde{X}) \longrightarrow \Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X}_1). \quad (22)$$

Hereafter, we denote by  $\Omega_3^{\text{rel}}(\tilde{X})$  this relative bordism  $\Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X}_1)$ , for simplicity.

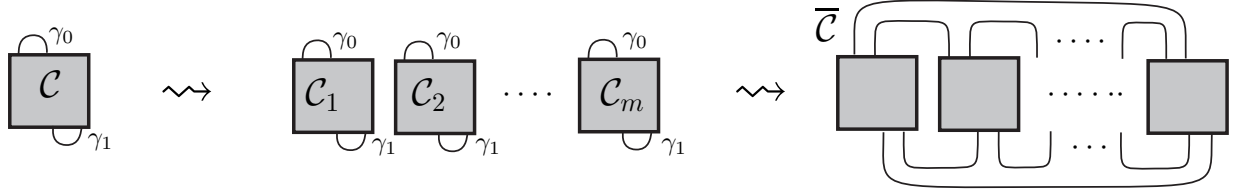


Figure 4: Construction of  $\overline{\mathcal{C}}$  from  $\mathcal{C}$ , when the link components of  $\mathcal{C}$  are two.

We now prove Proposition 6.13 using the following lemma:

**Lemma 6.14.** *For any connected quandle  $X$  of type  $t_X$ , the homomorphism  $\Upsilon_{\tilde{X}} : \Pi_2(\tilde{X}) \rightarrow \Omega_3^{\text{rel}}(\tilde{X})$  is surjective up to  $t_X$ -torsion.*

*Proof of Proposition 6.13.* We first explain the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_3(\text{As}(\tilde{X}))_{(\ell)} & \xrightarrow{\delta_*} & \Omega_3^{\text{rel}}(\tilde{X})_{(\ell)} & \longrightarrow & \Omega_2(B\tilde{X}_1)_{(\ell)} = 0 \\ & & \parallel & & \uparrow \Upsilon_{\tilde{X}} & & \\ 0 & \longrightarrow & H_3(\text{As}(\tilde{X}))_{(\ell)} & \xrightarrow{\tilde{\tau}_*} & \Pi_2(\tilde{X})_{(\ell)} & \longrightarrow & H_2^Q(\tilde{X})_{(\ell)} = 0 \end{array}$$

Here the top sequence is derived by the homology theory  $\Omega_n$  with considering the pair  $B\tilde{X}_1 \hookrightarrow K(\text{As}(\tilde{X}), 1)$ , and the bottom one is obtained from the  $P$ -sequence of  $\tilde{X}$  with Theorems 6.1 and 6.12. Hence, since  $\Pi_2(\tilde{X})$  is isomorphic to the finitely generated module  $\Omega_3(\text{As}(\tilde{X}))$  modulo  $t_X$  by assumption, the localized map of  $\Upsilon_{\tilde{X}}$  is an isomorphism by the previous lemma.

Therefore, to accomplish the proof, it suffices to show the equality

$$t_X \cdot (\delta_* \circ \Theta_{\tilde{X}}([\mathcal{C}])) = t_X \cdot \Upsilon_{\tilde{X}}([\mathcal{C}]) \in \Omega_3^{\text{rel}}(\tilde{X}), \quad (23)$$

for any  $\tilde{X}$ -coloring  $\mathcal{C}$ . For this, put the resulting link  $\overline{L}$  and coloring  $\overline{\mathcal{C}}$  explained above. Furthermore, take the  $t_X$ -fold covering  $p : C_{\overline{L}}^{t_X} \rightarrow S^3 \setminus \overline{L}$  and the natural inclusion  $i_{\mathcal{C}} : C_{\overline{L}}^{t_X} \subset \widehat{C}_{\overline{L}}^{t_X}$ . Here notice that the composite  $\theta_{\tilde{X},D}(\overline{\mathcal{C}}) \circ (i_{\mathcal{C}})_* : \pi_1(C_{\overline{L}}^{t_X}) \rightarrow \text{As}(X)$  is  $\Gamma_{\overline{\mathcal{C}}} \circ p_*$  by the definition (13). Furthermore, notice that the inclusion  $i_{\mathcal{C}}$  gives a bordance relation between the  $\theta_{\tilde{X},D}(\overline{\mathcal{C}})$  and this composite  $\theta_{\tilde{X},D}(\overline{\mathcal{C}}) \circ (i_{\mathcal{C}})_*$ . Since the above map  $\delta_*$  comes from the correspondences with  $(M, f)$  to  $(M, f)$  itself by definition, we thus have

$$t_X \cdot \delta_* \circ \Theta_{\tilde{X}}([\mathcal{C}]) = \delta_* \circ \Theta_{\tilde{X}}([\overline{\mathcal{C}}]) = [\theta_{\tilde{X},D}(\overline{\mathcal{C}})] = [\theta_{\tilde{X},D}(\overline{\mathcal{C}}) \circ (i_{\mathcal{C}})_*] = [\Gamma_{\overline{\mathcal{C}}} \circ p_*] \in \Omega_3^{\text{rel}}(\tilde{X}),$$

where the first equality is derived from  $t_X[\mathcal{C}] = [\overline{\mathcal{C}}] \in \Pi_2(\tilde{X})$  from the definition of  $\overline{L}$ . We notice  $[\Gamma_{\overline{\mathcal{C}}} \circ p_*] = t_X[\Gamma_{\overline{\mathcal{C}}}] \in \Omega_3^{\text{rel}}(\tilde{X})$  since the projection  $p$  takes the (relative) fundamental class of  $C_{\overline{L}}^{t_X}$  to the  $t_X$ -multiple of that of  $S^3 \setminus \overline{L}$ . Hence, we have the desired equality (23).  $\square$

To conclude this section, we will work out the proof of Lemma 6.14.

*Proof of Lemma 6.14.* To begin with, we claim that the  $\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$  is generated by bordism classes represented by (normal) 3-submanifolds in  $S^3$  with torus boundary components.

For this, we first explain the isomorphism (24) below. Here refer to the fact [Cla1, §2.5] that the universal covering of  $B\tilde{X}$  is a topological monoid; hence, it is a based loop space of some space  $\mathcal{L}_X$ . We therefore have two homotopy fibrations

$$\Omega\mathcal{L}_X \longrightarrow B\tilde{X} \xrightarrow{c_*} K(\text{As}(\tilde{X}), 1), \quad B\tilde{X} \xrightarrow{c_*} K(\text{As}(\tilde{X}), 1) \xrightarrow{\mathcal{P}_L} \mathcal{L}_X.$$

From the right  $\mathcal{P}_L$ , we have an isomorphism localized at  $\ell$ :

$$(\mathcal{P}_L)_* : \Omega_3(B\tilde{X}, K(\text{As}(\tilde{X}), 1))_{(\ell)} \cong \Omega_3(\mathcal{L}_X)_{(\ell)}. \quad (24)$$

However, since the  $\mathcal{L}_X$  is 2-connected by definition, the Hurewicz theorem  $\pi_3(\mathcal{L}_X) \cong \Omega_3(\mathcal{L}_X)$  implies that the  $\Omega_3(\mathcal{L}_X)$  is generated by representative maps  $S^3 \rightarrow \mathcal{L}_X$ . Since the map  $(\mathcal{P}_L)_*$  can be regard as a map coming from collapse of each boundaries of manifolds, this isomorphism (24) implies that generators of the  $\Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X})_{(\ell)}$  are derived from 3-submanifolds in  $S^3$ .

Next, so as to verify the claim above, consider the inclusions

$$S^1 \subset B\tilde{X}_1 \subset B\tilde{X} \xrightarrow{c} K(\text{As}(\tilde{X}), 1),$$

where the first is obtained by taking the circle labeled by  $a \in \tilde{X}$ . Noticing from Corollary 6.4 that these  $S^1 \subset B\tilde{X}_1 \subset K(\text{As}(\tilde{X}), 1)$  induce isomorphisms  $H_i(S^1)_{(\ell)} \cong H_i(B\tilde{X}_1)_{(\ell)} \cong H_i^{\text{gr}}(\text{As}(\tilde{X}))_{(\ell)} \cong 0$  for  $i = 2$ , they yield isomorphisms

$$\Omega_3(K(\text{As}(\tilde{X}), 1), S^1)_{(\ell)} \cong \Omega_3^{\text{rel}}(\tilde{X})_{(\ell)} \cong \Omega_3(K(\text{As}(\tilde{X}), 1), B\tilde{X})_{(\ell)}.$$

Here note that, since the last term is generated by classes from 3-manifolds in  $S^3$  as observed above and  $\pi_1(S^1)$  is abelian, the first term is generated by classes from 3-manifolds in  $S^3$  with torus boundaries<sup>3</sup>. Hence, so is the  $\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$  as claimed.

<sup>3</sup>To be precise, since the first homology of any closed surfaces is generated by homology classes from some tori, given a submanifold  $M \subset S^3$  with  $f : \pi_1(M) \rightarrow \text{As}(X)$  such that  $f(\pi_1(\partial M)) \subset \mathbb{Z}$ , we can obtain another  $M' \subset S^3$  with torus boundaries by attaching some 2-handles to  $M$ , and the  $f$  extend to  $f : \pi_1(M') \rightarrow \text{As}(X)$  such that  $f(\pi_1(\partial M')) \subset \mathbb{Z}$ .

Finally, to show the required surjectivity of  $\Upsilon_{\tilde{X}}$ , we will prove that any generator  $O$  of  $\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$  comes from some  $\tilde{X}$ -coloring. By the previous claim,  $t_X^{-2} \cdot O$  is represented by a homomorphism  $f : \pi_1(S^3 \setminus L) \rightarrow \text{As}(\tilde{X})$  for some links  $L \subset S^3$ . Furthermore, put the resulting link  $\bar{L}$  in the discussion in constructing  $\Upsilon_{\tilde{X}}$  in (22). By repeating process, we have another  $\bar{L}$ . Then the  $f$  extends to two  $\bar{f} : \pi_1(S^3 \setminus \bar{L}) \rightarrow \text{As}(\tilde{X})$  and  $\bar{\bar{f}} : \pi_1(S^3 \setminus \bar{\bar{L}}) \rightarrow \text{As}(\tilde{X})$  canonically, where note that the class of the latter in  $\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$  equals the  $O$ . Notice that, for each link component  $1 \leq i \leq \#L$ , with a choice of meridian  $\mathbf{m}_i$ , the  $\bar{f}(\mathbf{m}_i) \in \text{As}(\tilde{X})$  is conjugate to  $e_a^{n_i}$  for some  $n_i \in \mathbb{N}$ , from the definition of  $\Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$ : Namely, in the Wirtinger presentation of  $\pi_1(S^3 \setminus \bar{L})$ , each arc  $\alpha$  is labeled by  $e_{y_\alpha}^{n_i}$  for some  $y_\alpha \in \tilde{X}$ ; see (2). Accordingly, replacing the  $i$ -th component of the link  $\bar{L}$  by  $n_i$ -parallel copies of the component, we have another link  $\bar{L}_P$  and, then, construct a canonical homomorphism  $\bar{f}_P : \pi_1(S^3 \setminus \bar{L}_P) \rightarrow \text{As}(\tilde{X})$  such that each meridian of  $\bar{L}_P$  is sent to  $e_y$  for some  $y \in \tilde{X}$  (see Figure 5). We remark that, this  $\bar{f}_P$  sends the associated longitude  $\mathbf{l}_k$  of  $\bar{L}_P$  sends to  $e_y^{t_X n_y}$  for some  $n_y \in \mathbb{Z}$ , by the reason similar to the construction of  $\Upsilon_{\tilde{X}}$  in (22). In particular, we have  $y \cdot \bar{f}_P(\mathbf{l}_y) = y \cdot e_y^{t_X n_y} = y \in \tilde{X}$ . Hence, the bijection (4) admits an  $\tilde{X}$ -coloring  $\mathcal{C}_f$  such that  $\Gamma_{\mathcal{C}_f} = \bar{f}_P : \pi_1(S^3 \setminus \bar{L}_P) \rightarrow \text{As}(\tilde{X})$ . Consequently, we have the equality  $\Upsilon_{\tilde{X}}([\mathcal{C}_f]) = O \in \Omega_3^{\text{rel}}(\tilde{X})_{(\ell)}$  by definition, which implies the surjectivity.  $\square$

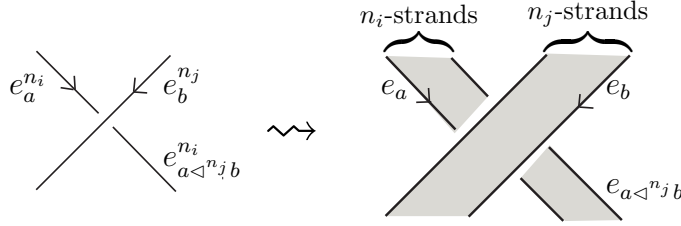


Figure 5: Construction from the  $\bar{f} : \pi_1(S^3 \setminus \bar{L}) \rightarrow \text{As}(\tilde{X})$  to  $\bar{f}_P : \pi_1(S^3 \setminus \bar{L}_P) \rightarrow \text{As}(\tilde{X})$ .

## 7 Proof of Theorems 3.10, 3.12.

We will compute the  $\Pi_2(X)$  with respect to symplectic and orthogonal quandles in §7.1, and that of connected quandles of order  $\leq 8$  in §7.2, as stated in Theorems 3.10, 3.12, respectively. The fundamental line of the proofs is to study the exact sequence in (21), which we call  $P$ -sequence. Actually, the proofs result from computation of the terms  $H_3^{\text{gr}}(\text{As}(X))$  and  $H_2^Q(X)$  concretely including these  $t_X$ -torsions.

### 7.1 On symplectic and orthogonal quandles over $\mathbb{F}_q$

We will prove Theorem 3.10 about computing the  $\Pi_2(X)$  for symplectic and orthogonal quandles  $X$  over finite fields. Our computation essentially relies on some works of Quillen and Friedlander, who had calculated group homologies of some groups of Lie type over  $\mathbb{F}_q$ . We list their works after our proof.

*Proof of Theorem 3.10.* Let  $q = p^d$  be odd and not exceptional, i.e.,  $d(p-1) < 6$ .

For (I), we will mention some homologies associated with the symplectic quandle  $X = \mathbf{Sp}_q^n$ . As is shown,  $\text{As}(X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q)$  (Proposition A.4); hence Theorems 7.2, 7.3 below tell us the group homology  $H_3^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}/(q^2 - 1)$ . In addition, when  $n \geq 2$  the second quandle homology  $H_2^Q(X)$  vanishes; see Proposition B.3. Hence the  $P$ -sequence (20) is reduced to be an epimorphism  $\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X) \rightarrow 0$ . Since  $X$  is of type  $p$ , for  $n \geq 2$ , Theorem 3.4 thus concludes an isomorphism  $\mathbb{Z}/(q^2 - 1) \rightarrow \Pi_2(X)$  as required.

Next we work out the case  $n = 2$ . Note that the second quandle homology  $H_2^Q(X)$  is  $(\mathbb{Z}/p)^d$ ; see Proposition B.3. Thereby the  $P$ -sequence (20) is rewritten in

$$\mathbb{Z}/(q^2 - 1) \longrightarrow \Pi_2(X) \longrightarrow (\mathbb{Z}/p)^d \longrightarrow 0. \quad (25)$$

Using Theorem 3.4 again, we have  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d$  as required.

For (II), we similarly deal with the spherical quandle  $X = S_q^n$  with  $n \geq 3$ . Note  $H_3^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}/(q^2 - 1)$  without 2-torsion (Example A.6). Furthermore, if  $n \geq 3$ ,  $H_2^Q(X)$  is an elementary abelian 2-group (Proposition B.2). Hence, the desired isomorphism  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1)$  follows from the  $P$ -sequence (20) and Theorem 3.4.

Finally, we deal with the case  $n = 2$ . By Proposition B.2, the quandle homology  $H_2^Q(X)$  is  $\mathbb{Z}/(q - \delta_q)$ , where  $\delta_q = \pm 1$  is according to  $q \equiv \pm 1 \pmod{4}$ . Thereby the  $P$ -sequence (20) is

$$\mathbb{Z}/(q^2 - 1) \longrightarrow \Pi_2(X) \longrightarrow \mathbb{Z}/(q - \delta_q) \longrightarrow 0 \quad (\text{mod } 2). \quad (26)$$

Using Theorem 3.4 again, we reach the goal  $\Pi_2(X) \cong \mathbb{Z}/(q^2 - 1) \oplus \mathbb{Z}/(q - \delta_q) \pmod{2}$ .  $\square$

As mentioned above, we review the group homologies of the symplectic groups  $Sp(2g, \mathbb{F}_q)$  and the orthogonal groups  $O(n; \mathbb{F}_q)$ . There is nothing new until the end of this subsection. We start recalling the homologies of  $Sp(2, \mathbb{F}_q)$  and  $O(3; \mathbb{F}_q)$ .

**Proposition 7.1.** *If  $p \neq 2$  and  $q \neq 3, 9$ , then the first and second homologies of  $Sp(2g; \mathbb{F}_q)$  vanish, i.e.,  $H_1^{\text{gr}}(Sp(2g, \mathbb{F}_q)) = H_2^{\text{gr}}(Sp(2g, \mathbb{F}_q)) \cong 0$ .*

*Furthermore the  $\ell$ -torsions of the third homology of  $Sp(2, \mathbb{F}_q)$  are expressed by*

$$H_3^{\text{gr}}(Sp(2, \mathbb{F}_q))_{(\ell)} \cong (\mathbb{Z}/(q^2 - 1))_{(\ell)}, \quad \text{for } \ell \neq p.$$

*On the other hand, the homologies  $H_1^{\text{gr}}(O(3, \mathbb{F}_q))$  and  $H_2^{\text{gr}}(O(3, \mathbb{F}_q))$  are annihilated by 2. Furthermore, the  $\ell$ -torsions of the third homology of  $O(3, \mathbb{F}_q)$  are expressed by*

$$H_3^{\text{gr}}(O(3, \mathbb{F}_q))_{(\ell)} \cong (\mathbb{Z}/(q^2 - 1))_{(\ell)}, \quad \text{for } \ell \neq p, 2.$$

*Proof.* See [FP, VIII. §4] or [Fri], noting the order  $|O(3, \mathbb{F}_q)| = 2q(q^2 - 1)$ .  $\square$

We moreover review the group homologies of  $Sp(2g; \mathbb{F}_q)$  and  $O(n, \mathbb{F}_q)$  as follows:

**Theorem 7.2** ([FP, Fri]). *Let  $q = p^d$  be odd. The inclusion  $Sp(2, \mathbb{F}_q) \hookrightarrow Sp(2n, \mathbb{F}_q)$  induces isomorphisms  $H_3^{\text{gr}}(Sp(2, \mathbb{F}_q)) \cong_{(\ell)} H_3^{\text{gr}}(Sp(2n, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/(q^2 - 1)$  localized at  $\ell \neq p$ . Furthermore, for  $n \geq 3$ , the inclusion  $O(3, \mathbb{F}_q) \hookrightarrow O(n, \mathbb{F}_q)$  induces isomorphisms  $H_3^{\text{gr}}(O(3, \mathbb{F}_q)) \cong_{(\ell)} H_3^{\text{gr}}(O(n, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/(q^2 - 1)$  localized at  $\ell \neq p, 2$ .*



*Proof.* According to [FP], the inclusions induces isomorphisms their cohomology with  $\mathbb{Z}/\ell$ -coefficients. Taking the limits as  $n \rightarrow \infty$ , their homologies  $H_3^{\text{gr}}(Sp(\infty, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/q^2 - 1$  and  $H_3^{\text{gr}}(O(\infty, \mathbb{F}_q)) \cong_{(\ell)} \mathbb{Z}/q^2 - 1$  are known [Fri, Theorem 1.7]. Hence, by Propositions 7.1, the induced maps on localized homologies are isomorphisms.  $\square$

Finally, we focus on these  $p$ -torsion parts, and state the vanishing theorem.

**Theorem 7.3** (Quillen, see [Fri, §4]). *Let  $q = p^d$  be odd. If  $d(p - 1) > 6$ , then the  $p$ -torsion parts  $H_3^{\text{gr}}(Sp(2n, \mathbb{F}_q))_{(p)}$  and  $H_3^{\text{gr}}(O(n + 2, \mathbb{F}_q))_{(p)}$  vanish for any  $n \geq 1$ . Furthermore, if  $n$  is enough large, the  $p$ -vanishing hold even for  $d(p - 1) \leq 6$ .*

**Remark 7.4.** As a result in [Fri, Corollary 1.8], the inclusions  $Sp(2n; \mathbb{F}_q) \hookrightarrow GL(2n; \mathbb{F}_q) \hookrightarrow GL(\infty; \mathbb{F}_q)$  induce the isomorphism between the third homology  $H_3^{\text{gr}}(Sp(2n, \mathbb{F}_q))$  and the Quillen  $K$ -group  $K_3(\mathbb{F}_q) \cong H_3^{\text{gr}}(GL(\infty; \mathbb{F}_q)) \cong \mathbb{Z}/(q^2 - 1)$ , if  $d(p - 1) > 6$  or  $n$  is enough large.

For instance, following [Fri, §4], we can see that, when  $q = p = 5$  and  $n \geq 7$ , the third homology  $H_3^{\text{gr}}(Sp(2n, \mathbb{F}_q))$  is  $\mathbb{Z}/(5^2 - 1) = \mathbb{Z}/24$ . We later use this result in §8.

## 7.2 Connected quandles of order $\leq 8$ (Proof of Theorem 3.12).

We will prove Theorem 3.12, which stated that, for connected quandles  $X$  of order  $\leq 8$ , the homomorphisms  $\Theta_X \oplus \mathcal{H}_X : \Pi_2(X) \rightarrow H_3^{\text{gr}}(\text{As}(X)) \oplus \text{Im}(\mathcal{H}_X)$  are isomorphisms. The proof is obtained by computing concretely  $\Pi_2(X)$  from the list of connected quandles of order  $\leq 8$ . In the proofs, we will often use the homomorphism  $\Theta_{\Pi\Omega} : \Pi_2(X) \rightarrow \Omega_3(\text{Ker}(\epsilon_X))$  in Lemma 4.2.

### 7.2.1 Connected Alexander quandles of order 4, 8

We first determine the  $\Pi_2(X)$  of the connected quandle of order 4 as follows:

**Proposition 7.5.** *If  $X$  is the Alexander quandle of the form  $\mathbb{Z}[T]/(2, T^2 + T + 1)$ , then  $\Pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$ .*

*Proof.* We first recall the fact [N1, Proposition 4.5] which says that  $\Pi_2(X)$  is either  $\mathbb{Z}/2 \oplus \mathbb{Z}/4$  or  $\mathbb{Z}/2 \oplus \mathbb{Z}/8$  and  $H_2^Q(X) \cong \mathbb{Z}/2$ . By Theorem 3.4, hence it suffices to show an isomorphism  $H_3^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}/8$  modulo 3. For this, note  $\text{As}(X) \cong Q_8 \rtimes \mathbb{Z}$ , where  $Q_8$  is the quaternion group of order 8 (see [N1, Lemma 4.8] for details). Noting  $\text{Type}(X) = 3$ , consider the quotient  $Q_8 \rtimes \mathbb{Z}/3$ , which is isomorphic to  $Sp(2; \mathbb{F}_3)$ . By Proposition 7.1 and the transfer, we thus have  $H_3^{\text{gr}}(\text{As}(X)) \cong H_3^{\text{gr}}(Sp(2; \mathbb{F}_3)) \cong \mathbb{Z}/8$  modulo 3-torsion as desired.  $\square$

**Remark 7.6.** We here note a relation between  $\Pi_2(X)$  and its quandle homologies. As is known,  $H_2^Q(X) \cong \mathbb{Z}/2$  and  $H_3^Q(X) \cong \mathbb{Z}/4$  [CJKLS, Remark 6.10]. Namely the summand  $\mathbb{Z}/8$  of  $\Pi_2(X)$  is evaluated not by the *quandle* cohomology, but by the *group* cohomology  $H_{\text{gr}}^3(Q_8; \mathbb{Z}/8)$ . It is therefore sensible to deal with 2-torsion of the groups  $\Pi_2(X)$  in general.

We next consider two Alexander quandles of order 8 of the forms  $X = \mathbb{Z}[T]/(2, T^3 + T^2 + 1)$  and  $X = \mathbb{Z}[T]/(2, T^3 + T + 1)$ . Then the both  $\Pi_2(X)$  were shown to be  $\mathbb{Z}/2$  [N2, Table 1].

We remark that, as is known [Cla3], a connected Alexander quandle  $X$  of order 4 or 8 is one of the three quandles above; hence the  $\Pi_2(X)$  is determined.

### 7.2.2 Two conjugate quandles of order 6

In this subsection, we calculate  $\Pi_2(X)$  of two quandles  $S_6, S'_6$  of order 6. Here, we define the quandle  $S_6$  (resp.  $S'_6$ ) to be the set of elements of a conjugacy class in the symmetric group  $\mathfrak{S}_4$  including  $(12) \in \mathfrak{S}_4$  (resp.  $(1234) \in \mathfrak{S}_4$ ) with the binary operation  $x \triangleleft y = y^{-1}xy$ .

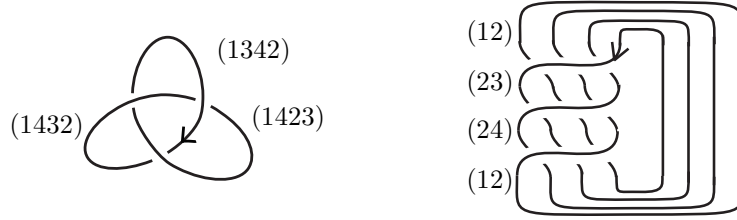


Figure 6: An  $S_6$ -coloring  $\mathcal{C}_3$  of the trefoil knot  $3_1$ , and an  $S'_6$ -coloring  $\mathcal{C}_4$  of  $T_{3,4}$ .

**Proposition 7.7.** *For the quandle  $S_6$ ,  $\Pi_2(S_6) \cong \mathbb{Z}/24 \oplus \mathbb{Z}/4$ . The first summand  $\mathbb{Z}/24$  is generated by  $\Xi_{S_6, 3_1}(\mathcal{C}_3)$ , where  $\mathcal{C}_3$  is a coloring of the trefoil knot shown as Figure 6.*

*On the other hand, for another quandle  $S'_6$ , we have  $\Pi_2(S'_6) \cong \mathbb{Z}/12$ . The generator is represented by  $\Xi_{S'_6, T_{3,4}}(\mathcal{C}_4)$ , where  $\mathcal{C}_4$  is a coloring of the torus knot  $T_{3,4}$  shown as Figure 6.*

*Proof.* We show the sequence (27). It follows from the proof of [N1, Proposition 4.9] that  $H_2^Q(S_6) \cong \mathbb{Z}/4$ , and further,  $H_3^{\text{gr}}(\text{As}(S_6))$  is a quotient of  $\mathbb{Z}/24$ , and  $H_2^{\text{gr}}(\text{As}(S_6)) \cong 0$ . Hence the  $P$ -sequence (21) becomes

$$\mathbb{Z}/24 \longrightarrow \Pi_2(S_6) \xrightarrow{\mathcal{H}} H_2^Q(X) (\cong \mathbb{Z}/4) \longrightarrow 0. \quad (27)$$

Next we will show that  $\Pi_2(S_6)$  surjects onto  $\mathbb{Z}/24$ . It is shown [N1, Lemma 4.10] that the kernel  $\text{Ker}(\epsilon_X)$  is the binary tetrahedral group  $D_{24} = Sp(2; \mathbb{F}_3)$  whose third homology is  $\mathbb{Z}/24$  (see Proposition 7.1). Let  $D_{24}$  act canonically on the sphere  $S^3$ . Since the 4-fold covering branched over the trefoil is  $S^3/D_{24}$  (see [Rol, §10.D]), it can be easily seen that the map  $\theta_{X,D}$  in (13) sends the  $X$ -coloring  $\mathcal{C}_3$  to the identity map  $\pi_1(S^3/D_{24}) \rightarrow Sp(2; \mathbb{F}_3)$ . Since  $\Omega_3(D_{24}) \cong \mathbb{Z}/24$  is known to be generated by the pair  $(S^3/D_{24}, \text{id}_{D_{24}})$  (see [Bro, VI. Examples 9.2]), the map  $\Phi_{\Pi\Omega} : \Pi_2(S_6) \rightarrow \Omega_3(D_{24})$  in Lemma 4.2 is surjective.

Finally, for proving the decomposition  $\Pi_2(S_6) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/24$ , it is enough to show that the class  $[\mathcal{C}_3]$  is sent to  $2 \in H_2^Q(X) \cong \mathbb{Z}/4$  by the Hurewicz map  $\mathcal{H}_X$ . By the formula (18), we have  $\mathcal{H}_X([\mathcal{C}_3]) = \Gamma_{\mathcal{C}_3}(\mathbf{I}) = e_{(1432)}^{-2} e_{(1432)} e_{(1423)} \in \text{Ker}(\epsilon_X) \cap \text{Stab}(x_0) \cong \mathbb{Z}/4$ . An elementary calculation can show the square  $\mathcal{H}([\mathcal{C}_3])^2 = 1$  and  $\mathcal{H}([\mathcal{C}_3]) \neq 1$  in  $\text{As}(X)$ , although we will not go into the details. In the sequel,  $\mathcal{H}([\mathcal{C}_3]) = 2 \neq 0$  as desired.

Changing the subject  $S'_6$ , we will compute  $\Pi_2(S'_6)$ . For this, we now show the sequence (28) below. It is shown [N1, Lemma 4.12, Appendix A.2] that  $H_2^Q(S_6) \cong \mathbb{Z}/2$ , and further,

the order  $|H_3^{\text{gr}}(\text{As}(S'_6))| \leq 12$ , and  $H_2^{\text{gr}}(\text{As}(S_6)) \cong \mathbb{Z}/2$ . Hence the  $P$ -sequence (21) becomes

$$H_3^{\text{gr}}(\text{As}(S'_6)) \longrightarrow \Pi_2(S'_6) \xrightarrow{\mathcal{H}} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \quad (28)$$

To complete the proof on  $\Pi_2(S'_6) \cong \mathbb{Z}/12$ , it is sufficient to show the surjectivity of the map  $\Theta_{\Pi\Omega} : \Pi_2(S'_6) \rightarrow \mathbb{Z}/12$ . As is shown [N1, Lemma 4.12], the kernel  $\text{Ker}(\epsilon_X)$  is the alternating group  $A_4$  of order 12 whose third homology is  $\mathbb{Z}/12$ . Since the quandle  $S'_6$  is of type 2 and the double covering space branched over the knot  $T_{3,4}$  is  $S^3/D_{24}$  (see [Rol, §10. D, E]), we can show that the map  $\theta_{X,D}$  in (13) sends the  $X$ -coloring  $\mathcal{C}_4$  to the epimorphism  $\pi_1(S^3/D_{24}) = D_{24} \rightarrow A_4$ , which is a central extension. Hence, the class  $[\theta_{X,D}(\mathcal{C}_4)]$  is a generator of  $\Omega_3(A_4) \cong \mathbb{Z}/12$ . This means the surjectivity  $\Theta_{\Pi\Omega} : \Pi_2(S'_6) \rightarrow \mathbb{Z}/12$ ; recalling  $|H_3^{\text{gr}}(\text{As}(S'_6))| \leq 12$  and the sequence (28) above, we have an isomorphism  $\Pi_2(S'_6) \cong \mathbb{Z}/12$ . Furthermore, by this process, the generator is presented by the coloring  $\mathcal{C}_4$ .  $\square$

### 7.2.3 The remaining quandle and the proof of Theorem 3.12

Finally, the rest of connected quandle of order 8 is the (extended) quandle  $\tilde{X}$  explained in §3.3, where  $X$  is the Alexander quandle of the form  $X = \mathbb{Z}[T]/(2, T^2 + T + 1)$ . Since  $\text{As}(X) \cong Q_8 \rtimes \mathbb{Z}$  (see the proof of Proposition 7.5), we have  $|\tilde{X}| = 8$  by definition.

**Proposition 7.8.** *Let  $\tilde{X}$  be the above quandle of order 8. Then  $\Pi_2(\tilde{X}) \cong \mathbb{Z}/8$ .*

*Proof.* We first show that the induced map  $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$  is an isomorphism. As is seen in the proof of Proposition 7.5, we note  $H_2^{\text{gr}}(\text{As}(X)) \cong 0$  and  $H_1^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}$ . Since the  $p_*$  is a central extension (see Proposition 6.8),  $p_*$  is an isomorphism.

Hence, we obtain  $H_2^Q(\tilde{X}) \cong 0$  from Theorem 6.12. Therefore, noting  $H_3^{\text{gr}}(\text{As}(\tilde{X})) \cong H_3^{\text{gr}}(Q_8 \rtimes \mathbb{Z}) \cong \mathbb{Z}/8$  from the proof of Proposition 7.7, the  $P$ -sequence is reduced to be an epimorphism  $\mathbb{Z}/8 \rightarrow \Pi_2(\tilde{X})$ . By Theorem 3.18, we conclude  $\Pi_2(\tilde{X}) \cong \mathbb{Z}/8$ .  $\square$

*Proof of Theorem 3.12.* One first deals with quandles  $X$  with  $|X| = 3, 5, 7$ . Then  $X$  is shown to be an Alexander quandles over the finite field  $\mathbb{F}_{|X|}$  [EGS]. By Theorem 3.9, the isomorphism  $\Theta_X \oplus \mathcal{H}_X$  holds for such  $X$ . On the other hand, a connected quandle of even order with  $|X| \leq 8$  is known to be one of the quandles in the previous subsections (see [Cla3]). Hence the proof completes.  $\square$

## 8 Dehn quandle of genus $\geq 7$

This section deals with Dehn quandles. To discuss this, we fix some notation:

**Notation** Denote by  $\Sigma_{g,k}$  the closed surface of genus  $g$  with  $k$  boundaries as usual. Let  $\mathcal{M}_{g,k}$  denote the mapping class group of  $\Sigma_{g,k}$  which is the identity on the  $k$ -boundaries. In the case  $k = 0$ , we often suppress the symbol  $k$ , e.g.,  $\Sigma_{g,0} = \Sigma_g$ .

We now review Dehn quandles [Y]. Consider the set,  $\mathcal{D}_g$ , defined by

$$\mathcal{D}_g := \{ \text{isotopy classes of (unoriented) non-separating simple closed curves } \gamma \text{ in } \Sigma_g \}.$$

For  $\alpha, \beta \in \mathcal{D}_g$ , we define  $\alpha \triangleleft \beta \in \mathcal{D}_g$  by  $\tau_\beta(\alpha)$ , where  $\tau_\beta \in \mathcal{M}_g$  is the positive Dehn twist along  $\beta$ . The pair  $(\mathcal{D}_g, \triangleleft)$  is a quandle, and called *(non-separating) Dehn quandle*. As is well-known, any two non-separating simple closed curves are related by some Dehn twists; the quandle  $\mathcal{D}_g$  is connected, and is not of any type  $t_X$ . In addition, since the Dehn twists are transvections in the view of the cohomology  $H^1(\Sigma_g; \mathbb{F}_p)$ , for any prime  $p$ , we have a quandle epimorphism  $\mathcal{P}_p$  from  $\mathcal{D}_g$  to the symplectic quandle  $\mathbf{Sp}_p^g$ , that is,  $\mathcal{P}_p : \mathcal{D}_g \rightarrow \mathbf{Sp}_p^g$ . The Dehn quandle  $\mathcal{D}_g$  is applicable to study 4-dimensional Lefschetz fibrations (see, e.g., [Y, Zab, N3]).

Our result is to determine the second homotopy groups  $\pi_2(B\mathcal{D}_g)$  in a stable range as follows:

**Theorem 8.1.** *Let  $g \geq 7$ . The homotopy group  $\pi_2(B\mathcal{D}_g)$  is isomorphic to either  $\mathbb{Z} \oplus \mathbb{Z}/24$  or  $\mathbb{Z} \oplus \mathbb{Z}/48$ . Furthermore, its torsion subgroup is generated by a  $\mathcal{D}_g$ -coloring in Figure 7.*

*Proof.* We first observe homologies of the associated group  $\text{As}(\mathcal{D}_g)$ . Note the well-known facts  $H_1^{\text{gr}}(\mathcal{M}_g) \cong 0$  and  $H_2^{\text{gr}}(\mathcal{M}_g) \cong \mathbb{Z}$  (see [FM]). Putting the universal central extension  $\mathcal{T}_g \rightarrow \mathcal{M}_g$ , Gervais [Ger] showed the isomorphism  $\text{As}(\mathcal{D}_g) \cong \mathbb{Z} \times \mathcal{T}_g$  (cf. Proposition A.3). Then Lemma 8.2 below and Kunneth theorem immediately imply isomorphisms  $H_2^{\text{gr}}(\text{As}(\mathcal{D}_g)) \cong 0$  and  $H_3^{\text{gr}}(\text{As}(\mathcal{D}_g)) \cong \mathbb{Z}/24$ .

We next study the  $P$ -sequences in respect to the above epimorphism  $\mathcal{P}_5 : \mathcal{D}_g \rightarrow \mathbf{Sp}_5^g$  with  $p = 5$ . Noting the isomorphism  $H_2^Q(\mathcal{D}_g) \cong \mathbb{Z}/2$  shown by [N3]. Furthermore recalling the vanishing  $H_2^Q(\mathbf{Sp}_5^g) \cong 0$  from Proposition B.3, these  $P$ -sequences are written in

$$\begin{array}{ccccccc} \mathbb{Z}/24 & \longrightarrow & \Pi_2(\mathcal{D}_g) & \longrightarrow & H_2^Q(\mathcal{D}_g) (\cong \mathbb{Z}/2) & \longrightarrow & 0 & \text{(exact)} \\ \downarrow (\mathcal{P}_5)_* & & \downarrow (\mathcal{P}_5)_* & & \downarrow (\mathcal{P}_5)_* & & & \\ H_3^{\text{gr}}(\text{As}(\mathbf{Sp}_5^g)) & \xrightarrow{\sim \delta_*} & \Pi_2(\mathbf{Sp}_5^g) (\cong \mathbb{Z}/24) & \longrightarrow & H_2^Q(\mathbf{Sp}_5^g) (\cong 0) & \longrightarrow & 0 & \text{(exact).} \end{array}$$

Here the proof of Theorem 3.10 says that the bottom  $\delta_*$  is an isomorphism  $H_3^{\text{gr}}(\text{As}(\mathbf{Sp}_5^g)) \rightarrow \Pi_2(\mathbf{Sp}_5^g) \cong \mathbb{Z}/24$ .

We next claim that the map  $(\mathcal{P}_5)_* : \Pi_2(\mathcal{D}_g) \rightarrow \Pi_2(\mathbf{Sp}_5^g)$  is surjective. By combing the homomorphism  $\Theta_{\Pi\Omega}$  in (14) with the epimorphism  $\mathcal{P}_5 : \mathcal{D}_g \rightarrow \mathbf{Sp}_5^g$  above, we have

$$\Pi_2(\mathcal{D}_g) \xrightarrow{(\mathcal{P}_5)_*} \Pi_2(\mathbf{Sp}_5^g) \xrightarrow{\Theta_{\Pi\Omega}} \Omega_3(Sp(2g; \mathbb{F}_5)). \quad (29)$$

To show the claim, it is enough to prove the surjectivity of this composite  $\Theta_{\Pi\Omega} \circ (\mathcal{P}_5)_*$ . Recall the isomorphisms  $\Omega_3(Sp(2g; \mathbb{F}_5)) \cong H_3^{\text{gr}}(Sp(2g; \mathbb{F}_5)) \cong \mathbb{Z}/24$  from Remark 7.4. Consider the  $\mathcal{D}_g$ -coloring  $\mathcal{C}$  illustrated below. Notice that the quandle  $\mathbf{Sp}_5^g$  is of type 5, and that 5-fold covering of  $S^3$  branched along the trefoil is the Poincaré sphere  $\Sigma(2, 3, 5)$  (see [Rol, §10.D]), whose  $\pi_1$  is  $Sp(2; \mathbb{F}_5)$  exactly. Using the map  $\theta_{X,D}$  in (13), we can see that the associated homomorphism  $\theta_{X,D}(\mathcal{C}) : \pi_1(\Sigma(2, 3, 5)) \rightarrow Sp(2; \mathbb{F}_5)$  is isomorphic. Hence, the class  $[\theta_{X,D}(\mathcal{C})]$  is a generator of  $\Omega_3(Sp(2; \mathbb{F}_5))$ . It follows from Theorem 7.2 that the inclusion  $Sp(2; \mathbb{F}_5) \hookrightarrow Sp(2g; \mathbb{F}_5)$  induces an isomorphism between these homologies without 5-torsion, which means the surjectivity.

However, the left vertical map  $(\mathcal{P}_5)_* : \mathbb{Z}/24 \rightarrow H_3^{\text{gr}}(\text{As}(\mathbf{Sp}_5^g))$  is not surjective (see Lemma 8.3 below). Hence, the purpose  $\Pi_2(\mathcal{D}_g)$  is either  $\mathbb{Z}/48$  or  $\mathbb{Z}/24$  by observing the above commutative diagram carefully.  $\square$

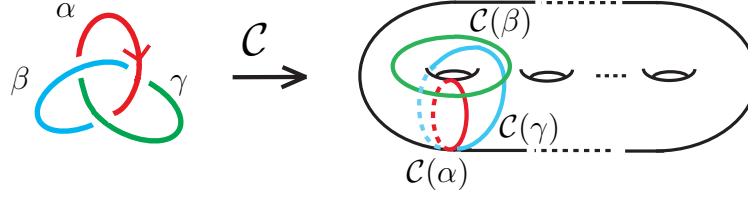


Figure 7: A  $\mathcal{D}_g$ -coloring of the trefoil knot

We now show two lemmas which are used in the proof above.

**Lemma 8.2.** *Let  $\mathcal{T}_g$  be the universal central extension on the group  $\mathcal{M}_g$ . If  $g \geq 3$ , then  $H_2^{\text{gr}}(\mathcal{T}_g)$  vanishes. Furthermore, if  $g \geq 7$ , then  $H_3^{\text{gr}}(\mathcal{T}_g) \cong \mathbb{Z}/24$ .*

We now prove Lemma 8.2 by using Quillen plus constructions and Madsen-Tillmann [MT].

*Proof.* We first immediately have  $H_2^{\text{gr}}(\mathcal{T}_g) \cong 0$ , since  $\mathcal{T}_g$  is the universal central extension of  $\mathcal{M}_g$  and the group  $\mathcal{M}_g$  is perfect (see, e.g., [Ros, Corollary 4.1.18]).

We next focus on  $H_3^{\text{gr}}(\mathcal{T}_g)$  with  $g \geq 3$ . Let  $B\mathcal{M}_{g,k}^+$  denote Quillen plus construction of Eilenberg-MacLane space of  $\mathcal{M}_{g,k}$  (see, e.g., [Ros, Chapter 5.2] for details). Since  $\mathcal{M}_{g,k}$  is perfect, the space  $B\mathcal{M}_{g,k}^+$  is simply connected. As a basic property of plus constructions (see [Ros, Theorem 5.2.7]), the homotopy group  $\pi_3(B\mathcal{M}_g^+)$  is isomorphic to  $H_3^{\text{gr}}(\mathcal{T}_g)$ .

Let us calculate  $\pi_3(B\mathcal{M}_g^+)$  for  $g \geq 7$ . For this, we set up some preliminaries. Consider the inclusion  $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g+1,1}$  obtained by gluing the surface  $\Sigma_{1,2}$  along one of its boundary components. Let  $\mathcal{M}_\infty := \lim_{g \rightarrow \infty} \mathcal{M}_{g,1}$ . Furthermore put an epimorphism  $\delta_g : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$  induced by gluing a disc to the boundary component of  $\Sigma_{g,1}$ . According to the Harer-Ivanov stability theorem improved by [RW], the inclusion  $\iota_\infty : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_\infty$  induces an isomorphism  $H_j^{\text{gr}}(\mathcal{M}_{g,1}) \cong H_j^{\text{gr}}(\mathcal{M}_\infty)$ , and the map  $\delta_g$  does  $H_j^{\text{gr}}(\mathcal{M}_g) \cong H_j^{\text{gr}}(\mathcal{M}_{g,1})$ , for  $j \leq 3$ .

We consider the maps  $\delta_g^+ : B\mathcal{M}_{g,1}^+ \rightarrow B\mathcal{M}_g^+$  and  $\iota_\infty^+ : B\mathcal{M}_{g,1}^+ \rightarrow B\mathcal{M}_\infty^+$  induced by  $\delta_g$  and  $\iota_\infty$ , respectively. By Whitehead theorem, these maps induce isomorphisms

$$(\delta_g^+)_* : \pi_3(B\mathcal{M}_{g,1}^+) \cong \pi_3(B\mathcal{M}_g^+), \quad (\iota_\infty^+)_* : \pi_3(B\mathcal{M}_{g,1}^+) \cong \pi_3(B\mathcal{M}_\infty^+).$$

However the  $\pi_3(B\mathcal{M}_\infty^+) \cong \mathbb{Z}/24$  was shown by Madsen and Tillmann [MT] (see also [Eb]). In summary, we have  $H_3^{\text{gr}}(\mathcal{T}_g) \cong \pi_3(B\mathcal{M}_\infty^+) \cong \mathbb{Z}/24$  as required.  $\square$

**Lemma 8.3.** *The induced map  $(\mathcal{P}_5)_* : H_3^{\text{gr}}(\text{As}(\mathcal{D}_g)) \rightarrow H_3^{\text{gr}}(\text{As}(\text{Sp}_5^g))$  with  $g \geq 7$  is not surjective.*

*Proof.* Recall the reduction of  $\mathcal{P}_5 : \text{As}(\mathcal{D}_g) \rightarrow \text{As}(\text{Sp}_5^g)$  to  $\mathbb{Z} \times \mathcal{T}_g \rightarrow \mathbb{Z} \times \text{Sp}(2g; \mathbb{F}_5)$ . We easily see that it factors through  $\mathbb{Z} \times \mathcal{M}_g$ . However  $H_3^{\text{gr}}(\mathcal{M}_g) \cong \mathbb{Z}/12$  is known [MT] (see also [Eb, §1]). Since  $H_3^{\text{gr}}(\mathcal{T}_g) \cong H_3^{\text{gr}}(\text{Sp}(2g; \mathbb{F}_5)) \cong \mathbb{Z}/24$  as above, the map  $(\mathcal{P}_5)_*$  is not a surjection.  $\square$

## 9 An application; third quandle homologies

As an application of the study of the homotopy group  $\pi_2(BX)$ , we compute some torsion subgroups of third quandle homologies  $H_3^Q(X)$  for finite connected quandles  $X$ . We first

prove Theorem 3.13 based on the facts explained in §9.1. In §9.2, we determine  $H_3^Q(X)$  for some quandles.

We briefly explain a basic line to study  $H_3^Q(X)$  in this section. Let  $B(X, X)$  be the rack space with respect to the primitive  $X$ -set. Note the following isomorphisms:

$$H_2(B(X, X)) \cong H_3^R(X) \cong H_3^Q(X) \oplus H_2^Q(X) \oplus \mathbb{Z}, \quad (30)$$

where the first isomorphism is derived from Remark 5.1, and the second was shown [LN, Theorem 2.2]. Composing this (30) with the result on  $\pi_2(BX) = \pi_2(B(X, X))$  from Theorem 3.4 can compute some torsion of the quandle homology  $H_3^Q(X)$  [see Lemma (9.4)].

Following this line, we first prove Theorem 3.13 as a general statement rewritten in

**Theorem 9.1** (Theorem 3.13). *Let  $X$  be a connected quandle with  $|X| < \infty$ . Let  $\text{Ker}(\psi_X)$  be the abelian kernel in (11). Then an isomorphism  $H_3^Q(X) \cong_{(\ell)} H_3^{\text{gr}}(\text{As}(X)) \oplus (\text{Ker}(\psi_X) \wedge \text{Ker}(\psi_X))$  holds after localization at any prime  $\ell$  which does not divide  $2|\text{Inn}(X)|/|X|$ .*

*Proof.* By Lemma 9.4 and the isomorphism (33) below, we have an isomorphism

$$\pi_2(BX)_{(\ell)} \oplus H_2^{\text{gr}}(\text{Ker}(\psi_X))_{(\ell)} \cong H_2(B(X, X))_{(\ell)}. \quad (31)$$

Recall from Theorem 3.4 the isomorphism  $\pi_2(BX)_{(\ell)} \cong H_3^{\text{gr}}(\text{As}(X))_{(\ell)} \oplus H_2^Q(X)_{(\ell)} \oplus \mathbb{Z}_{(\ell)}$ . Hence, together with (30) above, the isomorphism (31) is rewritten in

$$H_3^{\text{gr}}(\text{As}(X))_{(\ell)} \oplus H_2^Q(X)_{(\ell)} \oplus \mathbb{Z}_{(\ell)} \oplus H_2^{\text{gr}}(\text{Ker}(\psi_X))_{(\ell)} \cong \mathbb{Z}_{(\ell)} \oplus H_3^Q(X)_{(\ell)} \oplus H_2^Q(X)_{(\ell)}.$$

Since the second group homology  $H_2^{\text{gr}}(\text{Ker}(\psi_X))$  is the exterior product  $\bigwedge^2(\text{Ker}(\psi_X))$  [see [Bro, §V.6]], by a reduction of the both hand sides, we reach at the conclusion.  $\square$

## 9.1 Preliminaries

We now recall basic properties of the rack space  $B(X, Y)$  introduced in §6.1, which are used in the preceding proof. To begin, we review

**Proposition 9.2** ([FRS1, Theorem 3.7 and Proposition 5.1]). *Let  $X$  be a quandle, and  $Y$  an  $X$ -set. Decompose  $Y$  into the orbits as  $Y = \sqcup_{i \in I} Y_i$ . For  $i \in I$  and an element  $y_i \in Y_i$ , denote by  $\text{Stab}(y_i) \subset \text{As}(X)$  the stabilizer of  $y_i$ . Then, the subspace  $B(X, Y_i) \subset B(X, Y)$  is path-connected, and the natural projection  $B(X, Y_i) \rightarrow BX$  is a covering. Furthermore, the covering transformation group of the covering is the stabilizer  $\text{Stab}(y_i)$ .*

We next observe the spaces  $B(X, Y)$  in some cases of  $Y$ , where  $X$  is assumed to be connected. First, since  $\pi_1(BX) \cong \text{As}(X)$  from the 2-skeleton of  $BX$ , the projection  $B(X, Y) \rightarrow BX$  with  $Y = \text{As}(X)$  is the universal covering. Next, we let  $Y$  be the inner automorphism group  $\text{Inn}(X)$ , and be acted on by  $\text{As}(X)$  via (11). Considering the surjections  $\text{Inn}(X) \rightarrow X \rightarrow \{\text{pt}\}$  as  $X$ -sets, they then yield a sequence of the coverings

$$B(X, \text{Inn}(X)) \longrightarrow B(X, X) \longrightarrow BX. \quad (32)$$

**Remark 9.3.** According to Proposition 9.2, the covering transformation group of the second covering  $B(X, X) \rightarrow BX$  is the stabilizer  $\text{Stab}(x_0) \subset \text{As}(X)$ , and that of the composite  $B(X, \text{Inn}(X)) \rightarrow BX$  is the abelian kernel  $\text{Ker}(\psi_X)$  of  $\psi_X : \text{As}(X) \rightarrow \text{Inn}(X)$  in (11).

We further observe homologies of the space  $B(X, X)$ . Let  $\ell$  be a prime which does not divide the order  $|\text{Inn}(X)|/|X|$ . As is known, the action of  $\pi_1(BX)$  on the homology group  $H_*(BX)$  is trivial (see [Cla1]). Then the first covering in (32) induces an isomorphism between their homologies localized at  $\ell$ , that is,

$$H_*(B(X, \text{Inn}(X)))_{(\ell)} \cong H_*(B(X, X))_{(\ell)}. \quad (33)$$

Actually the transfer map of the covering provides an inverse (see [Cla1, Proposition 4.2]).

Finally we observe a relation between the homotopy and homology groups of the rack space  $B(X, \text{Inn}(X))$ . Refer to the fact that Clauwens [Cla1, §2.5] gave a topological monoid structure on  $B(X, \text{Inn}(X))$ . Hence

**Lemma 9.4.** *Let  $X$  be a connected quandle. Let  $BX_G$  denote the rack space  $B(X, \text{Inn}(X))$  for short. Then the Hurewicz homomorphism  $\pi_2(BX) = \pi_2(BX_G) \rightarrow H_2(BX_G)$  is a split injection modulo 2-torsion. In particular*

$$H_2(BX_G) \cong \pi_2(BX) \oplus H_2^{\text{gr}}(\text{Ker}(\psi_X)) \quad (\text{modulo 2-torsion}).$$

*Proof.* As is known, the second  $k$ -invariants of path-connected topological monoids with CW-structure are annihilated by 2 [Sou, AP]. Namely, the Hurewicz map  $\mathcal{H}$  splits modulo 2-torsion. Noting  $\pi_1(BX_G) \cong \text{Ker}(\psi_X)$  by Remark 9.3, we have  $\text{Coker}(\mathcal{H}) \cong H_2^{\text{gr}}(\text{Ker}(\psi_X))$ , which implies the required decomposition.  $\square$

## 9.2 Some calculations of third quandle homologies

We now prove Theorems 3.15, 3.17 based on the properties of rack spaces explained above: we will compute the third quandle homologies of some quandles in more details than Theorem 3.13 shown above. To begin, we will prove Theorem 3.15 in Alexander case.

*Proof of Theorem 3.15.* Let  $X$  be a regular Alexander quandle of finite order. Consider the rack space  $B(X, \text{Inn}(X))$  whose  $\pi_1$  is  $\text{Ker}(\psi_X) = \mathbb{Z} \times \mathcal{K}$  by Remark 9.3. It is shown that the torsion subgroups of the homology  $H_2(B(X, \text{Inn}(X)))$  and  $H_2(B(X, X))$  are annihilated by  $|X|^3$  (see [N2, Lemma 5.7] and [LN, Theorem 1.1]). Hence,  $H_3^Q(X)$  and  $H_3^{\text{gr}}(\text{As}(X))$  are annihilated by  $|X|^3$ , by repeating the proof of Theorem 9.1. Noticing  $t_X = |\text{Inn}(X)|/|X|$ , the purpose  $H_3^Q(X) \cong H_3^{\text{gr}}(\text{As}(X)) \oplus \mathcal{K} \oplus (\wedge^2 \mathcal{K})$  modulo 2 follows from the regularity and Theorem 9.1.

Finally, we work out the case where  $|X|$  is odd. By the above discussion, the homologies  $H_2(B(X, X))$  and  $H_2(B(X, \text{Inn}(X)))$  have no 2-torsion; so does the  $H_3^Q(X)$  as well.  $\square$

We next prove Theorem 3.17 to compute  $H_3^Q(X)$  for the symplectic and orthogonal quandles over  $\mathbb{F}_q$ , where we exclude the exceptional cases of  $q$ , i.e.,  $q \neq 3, 3^2, 3^3, 5, 7$ .

*Proof of Theorem 3.17.* (I) Let  $X = \mathbf{Sp}_q^n$  be the symplectic quandle over  $\mathbb{F}_q$ . Recall  $\text{As}(X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q)$  by Proposition A.4. We particularly see the kernel  $\text{Ker}(\psi_X) \cong \mathbb{Z}$ .

We deal with the case  $n = 1$ . Notice  $\text{Inn}(X) \cong Sp(2; \mathbb{F}_q)$  and  $|\text{Inn}(X)|/|X| = q$ . Therefore the purpose  $H_3^Q(X) \cong \mathbb{Z}/q^2 - 1$  without  $p$ -torsion follows from Theorem 3.13.

We now focus on the  $p$ -torsion of  $H_3^Q(X)$  with  $n = 1$ . For this, consider the rack space  $B(X, X)$  associated to the primitive  $X$ -set, whose  $P$ -sequence is given by

$$\rightarrow H_3^{\text{gr}}(\text{Stab}(x_0)) \rightarrow \pi_2(BX) \xrightarrow{\mathcal{H}} H_2(B(X, X)) \rightarrow H_2^{\text{gr}}(\text{Stab}(x_0)) \rightarrow 0.$$

We easily see the stabilizer  $\text{Stab}(x_0) \cong \mathbb{Z} \times (\mathbb{Z}/p)^d$  by Proposition A.4. Since  $\pi_2(BX) \cong \mathbb{Z} \oplus \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d$  by Theorem 3.10, the sequence is rewritten in

$$H_3^{\text{gr}}(\mathbb{Z} \times (\mathbb{Z}/p)^d) \rightarrow \mathbb{Z} \oplus \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^d \xrightarrow{\mathcal{H}} H_2(B(X, X)) \rightarrow H_2^{\text{gr}}(\mathbb{Z} \times (\mathbb{Z}/p)^d) \rightarrow 0. \quad (34)$$

We here claim that the Hurewicz map  $\mathcal{H}_{(p)}$  is a split injection. Indeed, by (25), the composite  $P_* \circ \mathcal{H} : \pi_2(BX) \rightarrow H_2(BX) \cong \mathbb{Z} \oplus (\mathbb{Z}/p)^d$  is surjective, where  $P : B(X, X) \rightarrow BX$  is the covering. Consequently, by this claim, the sequence above means

$$H_2(B(X, X)) \cong_{(p)} \pi_2(BX) \oplus H_2^{\text{gr}}(\mathbb{Z} \times (\mathbb{Z}/p)^d) \cong_{(p)} \mathbb{Z}_{(p)} \oplus (\mathbb{Z}/p)^d \oplus (\mathbb{Z}/p)^{d(d+1)/2}.$$

Hence, compared with the isomorphism (30), we have  $H_3^Q(X) \cong_{(p)} (\mathbb{Z}/p)^{d(d+1)/2}$  as desired.

Next, when  $n \geq 2$ , we will show  $H_3^Q(X) = 0$ . Note that the second group homology of the stabilizer  $\text{Stab}(x_0) = \pi_1(B(X, X))$  is zero (Lemma B.4). Furthermore, we note  $H_2(BX) \cong \mathbb{Z}$  by Proposition B.3. Thereby, the  $P$ -sequences of the projection  $P : B(X, X) \rightarrow BX$  is written in

$$\begin{array}{ccccccc} H_3^{\text{gr}}(\text{Stab}(x_0)) & \xrightarrow{\delta_*} & \pi_2(B(X, X)) & \longrightarrow & H_2(B(X, X)) & \longrightarrow & 0 \\ P_* \downarrow & & P_* \downarrow \cong & & P_* \downarrow & & \\ H_3^{\text{gr}}(\mathbb{Z} \times Sp(n; \mathbb{F}_q)) & \longrightarrow & \pi_2(BX) & \longrightarrow & H_2(BX) & (\cong \mathbb{Z}) & \longrightarrow 0. \end{array}$$

By the stability theorem 7.2, the left vertical map  $P_*$  surjects onto  $\mathbb{Z}/(q^2 - 1)$ . Since  $\pi_2(BX) \cong \mathbb{Z} \oplus \mathbb{Z}/(q^2 - 1)$  by Theorem 3.10, the delta map  $\delta_*$  is surjective in torsion parts. Therefore  $H_2(B(X, X)) = \mathbb{Z}$  by diagram chasing. Using (30) again, we have the goal  $H_3^Q(X) = 0$ .

(II) The calculations of the third homology  $H_3^Q(X)$  for the spherical quandle  $X = S_q^n$  over  $\mathbb{F}_q$  can be shown in a similar way to the symplectic case. The point is that the homology  $H_i^{\text{gr}}(\text{Stab}(x_0))$  is isomorphic to  $H_i^{\text{gr}}(O(n; \mathbb{F}_q))$  modulo 2-torsion for  $i \leq 3$  (see Proposition 7.1). So, using results of Quillen explained in §7.1, we can complete the proof similarly, and omit the details. Incidentally, the results of  $H_2^Q(X)$  follow from Proposition B.2.  $\square$

Moreover we will prove Theorem 3.18 on extended quandles.

*Proof of Theorem 3.18.* We first construct the claimed isomorphism  $\Pi_2(\tilde{X}) \cong H_3^Q(\tilde{X})$  modulo  $t_X$ . Consider the rack space  $B(\tilde{X}, \tilde{X})$  with respect to the primitive  $\tilde{X}$ -set. Note that this  $\pi_1$  is



an abelian group  $\mathbb{Z} \times \text{Ker}(p_*)$  by Remark 9.3 and Proposition 6.11. Then its Postnikov tower is expressed by

$$H_3^{\text{gr}}(\mathbb{Z} \times \text{Ker}(p_*)) \longrightarrow \pi_2(B\tilde{X}) \xrightarrow{\mathcal{H}_{\tilde{X}}} H_2(B(\tilde{X}, \tilde{X})) \longrightarrow H_2^{\text{gr}}(\mathbb{Z} \times \text{Ker}(p_*)) \rightarrow 0 \quad (\text{exact}).$$

By Proposition 6.11 again,  $H_i^{\text{gr}}(\mathbb{Z} \times \text{Ker}(p_*))$  is annihilated by  $t_X$ . Hence, the Hurewicz map  $\mathcal{H}_{\tilde{X}}$  is the isomorphism  $\Pi_2(\tilde{X}) \cong H_3^Q(\tilde{X})$  modulo  $t_X$ .

To prove the latter part, we note  $H_2^Q(\tilde{X}) \cong 0 \pmod{t_X}$  by Theorem 6.12. Considering the isomorphisms  $p_* : H_3^{\text{gr}}(\text{As}(\tilde{X})) \rightarrow H_3^{\text{gr}}(\text{As}(X))$  and  $\Theta_X : \Pi_2(\tilde{X}) \rightarrow H_3^{\text{gr}}(\text{As}(\tilde{X}))$  modulo  $t_X$  in Proposition 6.10 and Theorem 3.4 respectively, we obtain the composite isomorphism  $p_* \circ \Theta_X : \Pi_2(\tilde{X}) \rightarrow H_3^{\text{gr}}(\text{As}(X))$  modulo  $t_X$  as desired.  $\square$

Finally, we will show a lemma which will be used in a subsequent paper [N4].

**Lemma 9.5.** *Let  $X$  be a connected Alexander quandle of type  $t_X$ . Let  $p : \tilde{X} \rightarrow X$  be the projection. Then the induced map  $p_* : H_3^Q(\tilde{X}) \rightarrow H_3^Q(X)$  is an injection up to  $2t_X$ -torsion.*

*Proof.* By the proof of Theorem 3.15 in §9.2, the Hurewicz map  $\mathcal{H}_X : \pi_2(B(X, X)) \rightarrow H_2(B(X, X))$  is injective up to  $2t_X$ -torsion. Again consider the Postnikov tower with respect to the  $p : \tilde{X} \rightarrow X$ :

$$\begin{array}{ccccccc} H_3^{\text{gr}}(\mathbb{Z} \times \text{Ker}(p_*))_{(\ell)} & \xrightarrow{0} & \pi_2(B\tilde{X})_{(\ell)} & \xrightarrow{\mathcal{H}_{\tilde{X}}} & H_2(B(\tilde{X}, \tilde{X}))_{(\ell)} & \longrightarrow & H_2^{\text{gr}}(\mathbb{Z} \times \text{Ker}(p_*))_{(\ell)} = 0 \\ p_* \downarrow & & p_* \downarrow & & p_* \downarrow & & p_* \downarrow \\ H_3^{\text{gr}}(\text{Stab}(x_0))_{(\ell)} & \xrightarrow{0} & \pi_2(BX)_{(\ell)} & \xrightarrow{\mathcal{H}_X} & H_2(B(X, X))_{(\ell)} & \longrightarrow & H_2^{\text{gr}}(\text{Stab}(x_0))_{(\ell)}. \end{array}$$

Here a prime  $\ell$  is relatively prime to  $2t_X$ . Since the top Hurewicz map  $\mathcal{H}_{\tilde{X}}$  is an isomorphism modulo  $2t_X$  (see the previous proof of Theorem 3.18), the vertical map  $p_* : H_2(B(\tilde{X}, \tilde{X}))_{(\ell)} \rightarrow H_2(B(X, X))_{(\ell)}$  is injective. Hence, by (30) as usual, the  $p_* : H_3^Q(\tilde{X}) \rightarrow H_3^Q(X)$  turns out to be an injection up to  $2t_X$ -torsion.  $\square$

## A Calculations of automorphism groups of quandles

According to Theorem 3.4, it is significant to determine associated groups  $\text{As}(X)$ . In this appendix, we develop a method to formulate  $\text{As}(X)$  concretely from the inner automorphism groups  $\text{Inn}(X)$ , using the group extension (11) which is rewritten in

$$0 \longrightarrow \text{Ker}(\psi_X) \longrightarrow \text{As}(X) \xrightarrow{\psi_X} \text{Inn}(X) \longrightarrow 0 \quad (\text{central extension}). \quad (35)$$

Here, we note that, when  $X$  is connected and of type  $t_X$ , the kernel is isomorphic to  $\mathbb{Z} \oplus H_2^{\text{gr}}(\text{Inn}(X))$  up to  $t_X$ -torsion (Corollary 6.4). In conclusion, to investigate  $\text{As}(X)$ , we shall study the  $\text{Inn}(X)$  and  $H_2^{\text{gr}}(\text{Inn}(X)) \pmod{t_X}$ , metaphorically speaking, ‘universal central extensions’ of  $\text{Inn}(X)$  modulo  $t_X$ -torsion.

To study  $\text{As}(X)$ , we first propose a simple method of determining the group  $\text{Inn}(X)$ .

**Lemma A.1.** *Let a group  $G$  act on a quandle  $X$ . Let a map  $\kappa : X \rightarrow G$  satisfy the followings:*

- (i) The identity  $x \triangleleft y = x \cdot \kappa(y) \in X$  holds for any  $x, y \in X$ .
- (ii) The identity  $\kappa(x)\kappa(y) = \kappa(y)\kappa(x \triangleleft y) \in G$  holds for any  $x, y \in X$ .
- (iii) The image  $\kappa(X) \subset G$  generates the group  $G$ , and the action  $X \curvearrowright G$  is effective.

Then there is an isomorphism  $\text{Inn}(X) \cong G$ , and the action  $X \curvearrowright G$  coincides with the action of  $\text{Inn}(X)$ .

*Proof.* Identifying the action  $X \curvearrowright G$  with a group homomorphism  $F : G \rightarrow \mathfrak{S}_{|X|}$ , this  $F$  factors through  $\text{Inn}(X)$  by (i). Furthermore, (ii) implies that the epimorphism  $\psi_X$  in (35) is decomposed as  $\text{As}(X) \rightarrow G \xrightarrow{F} \text{Inn}(X)$ . Moreover (iii) concludes the bijectivity of  $F : G \cong \text{Inn}(X)$  and, hence, the coincidence of the two actions by construction.  $\square$

This lemma is applicable and practical; actually, for many quandles  $X$ , we write down the inner automorphism groups  $\text{Inn}(X)$  in this way. For example, we now deal with the symplectic quandles  $\mathbf{Sp}_K^n$  and spherical quandles  $S_K^n$  over  $K$  (Examples 2.2, 2.3) as follows:

**Lemma A.2.** *Let  $K$  be a commutative field of characteristic not equal to 2. Then  $\text{Inn}(\mathbf{Sp}_K^n)$  is isomorphic to the symplectic group  $Sp(2n; K)$ . Furthermore, if  $n \geq 2$ , then  $\text{Inn}(S_K^n)$  is isomorphic to the orthogonal group  $O(n+1; K)$ .*

*Proof.* As is called the Cartan-Dieudonné theorem classically, the groups  $Sp(2n; K)$  and  $O(n+1; K)$  are generated by transvections and symmetries ( $\bullet \triangleleft y$ ), respectively.

We will show the isomorphism  $\text{Inn}(\mathbf{Sp}_K^n) \cong Sp(2n; K)$ . For any  $y \in \mathbf{Sp}_K^n$ , the map  $(\bullet \triangleleft y) : \mathbf{Sp}_K^n \rightarrow \mathbf{Sp}_K^n$  is a restriction of a linear map  $K^{2n} \rightarrow K^{2n}$ . It thus yields a map  $\kappa : \mathbf{Sp}_K^n \rightarrow GL(2n; K)$ , which factors through the  $Sp(2n; K)$  and satisfies the conditions in Lemma A.1. Indeed, e.g., the condition (iii) follows from the classical theorem and the effectivity of the standard action  $K^{2n} \curvearrowright Sp(2n; K)$ . Therefore  $\text{Inn}(\mathbf{Sp}_K^n) \cong Sp(2n; K)$  as desired.

Turning to the orthogonal case, the isomorphism  $\text{Inn}(S_K^n) \cong O(n+1; K)$  can be shown by replacing  $\mathbf{Sp}_K^{2n}$  by  $S_K^n$  and  $Sp(2n; K)$  by  $O(n+1; K)$ , respectively, in the previous proof.  $\square$

Next, using the extension (35), we calculate the associated groups of some quandles.

**Proposition A.3.** *Put the epimorphism  $\epsilon_X : \text{As}(X) \rightarrow \mathbb{Z}$  in (3). If  $X$  is connected and  $\text{Inn}(X)$  is perfect, i.e.,  $H_1^{\text{gr}}(\text{Inn}(X)) = 0$ , then we have an isomorphism  $\text{As}(X) \cong \text{Ker}(\epsilon_X) \times \mathbb{Z}$ , and  $\text{Ker}(\epsilon_X)$  is a central extension of  $\text{Inn}(X)$  and is perfect. In particular, if the second integral group homology of  $\text{Inn}(X)$  vanishes, then  $\text{As}(X) \cong \text{Inn}(X) \times \mathbb{Z}$ .*

*Proof.* We will show  $\text{As}(X) \cong \mathbb{Z} \times \text{Ker}(\epsilon_X)$ . Since  $H_1^{\text{gr}}(\text{Inn}(X)) = 0$ , we obtain an epimorphism  $\text{Ker}(\psi_X) \hookrightarrow \text{As}(X) \xrightarrow{\text{proj}} H_1^{\text{gr}}(\text{As}(X)) = \mathbb{Z}$  from (35). We can choose a section  $\mathfrak{s} : \mathbb{Z} \rightarrow \text{As}(X)$ . Hence, by the equality (2), the semi-product  $\text{As}(X) \cong \text{Ker}(\epsilon_X) \rtimes \mathbb{Z}$  is trivial. Namely,  $\text{As}(X) \cong \text{Ker}(\epsilon_X) \times \mathbb{Z}$  as desired. Furthermore, by construction, the kernel  $\text{Ker}(\epsilon_X)$  is a central extension of  $\text{Inn}(X)$ , and is perfect by the Kunneth theorem and  $\text{As}(X)_{\text{ab}} \cong \mathbb{Z}$ .  $\square$

**Example A.4.** Let  $q \neq 3, 9$ . Let  $X$  be the symplectic quandles  $\mathbf{Sp}_q^n$  over  $\mathbb{F}_q$ . Then we see  $\text{As}(X) \cong \mathbb{Z} \times Sp(2n; \mathbb{F}_q)$ . In fact, noticing  $\text{Inn}(X) \cong Sp(2n; \mathbb{F}_q)$  from Lemma A.2, the first and second homologies of  $Sp(2n; \mathbb{F}_q)$  vanish (see Proposition 7.1 or [FP, Fri]).

In general, it is hard to calculate the associated groups  $\text{As}(X)$  concretely; however, in some cases, we can calculate some torsion parts of their group homologies as follows:

**Lemma A.5.** *Let  $X$  be a connected quandle of type  $t_X$ . If  $H_2^{\text{gr}}(\text{Inn}(X))$  is annihilated by  $t_X$ , then there is an isomorphism  $H_3^{\text{gr}}(\text{As}(X)) \cong H_3^{\text{gr}}(\text{Inn}(X))$  modulo  $t_X$ -torsion.*

*Proof.* By Corollary 6.4 we have  $\text{Ker}(\psi_X) \cong \mathbb{Z} \pmod{t_X}$ . Hence, the Lyndon-Hochschild spectral sequence of the  $\psi_X$  leads to the required isomorphism modulo  $t_X$ -torsion.  $\square$

**Example A.6.** Let  $n \geq 2$ , and  $q \neq 3, 9$ . Let  $X = S_q^n$  be the spherical quandle over  $\mathbb{F}_q$  of type 2. Then  $H_3^{\text{gr}}(\text{As}(X)) \cong H_3^{\text{gr}}(O(n+1; \mathbb{F}_q))$  without 2-torsion. Actually, the  $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(O(n+1; \mathbb{F}_q))$  is known to be annihilated by 2 (see Theorems 7.2, 7.3).

Incidentally, we compare the type with the order  $|\text{Inn}(X)|/|X|$  as follows:

**Lemma A.7.** *Let  $X$  be a finite connected quandle. Then its type  $t_X$  is a divisor of  $|\text{Inn}(X)|/|X|$ .*

*Proof.* For  $x, y \in X$ , we define  $m_{x,y}$  by the minimal  $n$  satisfying  $x \triangleleft^n y = x$ . Note that  $(\bullet \triangleleft^{m_{x,y}} y)$  lies in the stabilizer  $\text{Stab}(x)$ . Since  $|\text{Stab}(x)| = |\text{Inn}(X)|/|X|$  by connectivity,  $m_{x,y}$  divides  $|\text{Inn}(X)|/|X|$ ; hence so does the type  $t_X$ .  $\square$

## B Some calculations of second quandle homologies

This appendix calculates the second quandle homologies of some connected quandles  $X$ , using the results on  $\text{As}(X)$  in the previous section. Our calculation relies on a result of Eisermann [E2] (see Theorem 5.2) which claims an isomorphism

$$H_2^Q(X) \cong (\text{Ker}(\epsilon_X) \cap \text{Stab}(x_0))_{\text{ab}}.$$

We first consider the  $H_2^Q(X)$  for a connected Alexander quandle  $X$ . Clauwens [Cla2] determined the associated group  $\text{As}(X)$  as follows. Set up a homomorphism  $\mu_X : X \otimes X \rightarrow X \otimes X$  defined by  $\mu_X(x \otimes y) = x \otimes y - Ty \otimes x$ . He defined a group operation on  $\mathbb{Z} \times X \times \text{Coker}(\mu_X)$  by setting

$$(n, x, \alpha) \cdot (m, y, \beta) = (n + m, T^m x + y, \alpha + \beta + [T^m x \otimes y]),$$

and showed that the homomorphism  $\text{As}(X) \rightarrow \mathbb{Z} \times X \times \text{Coker}(\mu_X)$  defined by sending  $e_x$  to  $(1, x, 0)$  is a group isomorphism. We then describe the lower central series of  $\text{As}(X)$  as

$$\text{As}(X) \supset X \times \text{Coker}(\mu_X) \supset \text{Coker}(\mu_X) \supset 0. \quad (36)$$

As a result, we see that the kernel of  $\psi_X : \text{As}(X) \rightarrow \text{Inn}(X)$  equals  $\mathbb{Z} \times \text{Coker}(\mu_X)$ . Thanks to his presentation of  $\text{As}(X)$ , we easily show a result of Clauwens:

**Proposition B.1** (Clauwens [Cla2]). *Let  $X$  be a connected Alexander quandle. The homology  $H_2^Q(X)$  is isomorphic to the quotient module  $\text{Coker}(\mu_X) = X \otimes_{\mathbb{Z}} X / (x \otimes y - Ty \otimes x)_{x,y \in X}$ .*

*Proof.* By definition we can see that the  $\text{Ker}(\epsilon_X) \cap \text{Stab}(0)$  is the cokernel  $\text{Coker}(\mu_X)$ .  $\square$

Next, we focus on second homologies of spherical and symplectic quandles over  $\mathbb{F}_q$  as follows.

**Proposition B.2.** *Let  $X = S_q^n$  be a spherical quandle over  $\mathbb{F}_q$ . Let  $q \neq 3, 9$ . For  $n \geq 3$ , the second homology  $H_2^Q(X)$  is an elementary abelian 2-group. If  $n = 2$ , then the homology  $H_2^Q(X)$  is the cyclic group  $\mathbb{Z}/(q - \delta_q)$  modulo 2-torsion, where  $\delta_q = \pm 1$  is according to  $q \equiv \pm 1 \pmod{4}$ .*

*Proof.* Under the standard action  $X \curvearrowright O(n+1; \mathbb{F}_q)$ , the stabilizer of  $(1, 0, \dots, 0) \in X$  is easily shown to be  $O(n; \mathbb{F}_q)$ . By a similar discussion to the proof of Proposition A.5, it follows from Theorem 5.2 that  $H_2^Q(X) \cong H_1^{\text{gr}}(O(n; \mathbb{F}_q))$  without 2-torsion. For  $n \geq 3$ , the abelianization of  $O(n-1; \mathbb{F}_q)$  is  $(\mathbb{Z}/2)^2$  [see [FP, II. §3]]. On the other hand, when  $n = 2$ , the group  $O(2; \mathbb{F}_q)$  is cyclic and of order  $q - \delta_q$ . Hence,  $H_2^Q(X) \cong H_1^{\text{gr}}(O(2; \mathbb{F}_q)) \cong \mathbb{Z}/(q - \delta_q) \pmod{2}$ .  $\square$

**Proposition B.3.** *Let  $X$  be the symplectic quandle  $\text{Sp}_q^n$  over  $\mathbb{F}_q$ . Let  $q \neq 3, 9$ . If  $n \geq 2$ , the second homology  $H_2^Q(X)$  vanishes. If  $n = 1$ , then  $H_2^Q(X) \cong (\mathbb{Z}/p)^d$ , where  $q = p^d$ .*

*Proof.* Recall  $\text{As}(X) \cong \mathbb{Z} \times \text{Sp}(2n; \mathbb{F}_q)$  from Example A.4. Considering the action  $X \curvearrowright \text{Sp}(2n; \mathbb{F}_q)$ , denote by  $G$  the stabilizer of  $(1, 0, \dots, 0) \in (\mathbb{F}_q)^{2n}$ . Since Theorem 5.2 immediately means  $H_2^Q(X) \cong H_1^{\text{gr}}(G)$ , we will calculate  $H_1^{\text{gr}}(G)$  as follows. First, for  $n = 1$ , it can be verified that the  $G$  is exactly the product  $(\mathbb{Z}/p)^d$  as an abelian group; hence  $H_2^Q(X) \cong (\mathbb{Z}/p)^d$  in the sequel. Next, for  $n \geq 2$ , the vanishing  $H_2^Q(X) = H_1^{\text{gr}}(G) = 0$  immediately follows from Lemma B.4 below.  $\square$

**Lemma B.4.** *Let  $n \geq 2$ . Let  $G$  denote the stabilizer of the action  $X = \text{Sp}_q^n \curvearrowright \text{Sp}(2n; \mathbb{F}_q)$  mentioned above. Then the homologies  $H_1^{\text{gr}}(G)$  and  $H_2^{\text{gr}}(G)$  vanish.*

*Proof.* Recall from [FP, II. §6.3] the order of  $\text{Sp}(2n; \mathbb{F}_q)$  as

$$|\text{Sp}(2n; \mathbb{F}_q)| = q^{n^2} (q^{2n} - 1)(q^{2n-2} - 1) \cdots (q^2 - 1).$$

Since  $|X| = q^{2n} - 1$ , the order of  $G$  is  $q^{2n-1} \cdot |\text{Sp}(2n-2; \mathbb{F}_q)|$ . Thereby  $H_1^{\text{gr}}(G)$  and  $H_2^{\text{gr}}(G)$  are zero modulo  $p$ -torsion, it is because of the inclusion  $\text{Sp}(2n-2; \mathbb{F}_q) \subset G$  by definitions and the vanishing  $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(\text{Sp}(2n-2; \mathbb{F}_q)) \cong 0$  modulo  $p$ .

Finally, we may focus on the  $p$ -torsion of  $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(G)$ . Following the proof of [Fri, Proposition 4.4], there is a certain subgroup “ $\Delta(\text{Sp}(2n; \mathbb{F}_q))$ ” of  $G$  which contains a  $p$ -sylog group of  $\text{Sp}(2n; \mathbb{F}_q)$ , and this  $\mathbb{Z}/p$ -homology vanishes. Hence  $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(G) = 0$  as required.  $\square$

## B.1 Proof of Theorem 5.2

We will show Theorem 5.2 as a result of Proposition B.5. This proposition provides an algorithm to compute the first rack homology as follows:

**Proposition B.5.** *Let  $X$  be a quandle, and  $Y$  an  $X$ -set. Decompose  $Y$  into the orbits as  $Y = \sqcup_{i \in I} Y_i$ . For  $i \in I$ , choose an arbitrary element  $y_i \in Y_i$ , and denote by  $\text{Stab}(y_i) \subset \text{As}(X)$  the stabilizer subgroup of  $y_i$ . Then the direct sum of the abelianizations of  $\text{Stab}(y_i)$  is isomorphic to  $H_1^R(X, Y)$ . That is,  $\oplus_{i \in I} (\text{Stab}(y_i))_{\text{ab}} \cong H_1^R(X, Y)$ .*

*Proof.* We investigate the first homology of the space  $B(X, Y_i)$  for each  $i \in I$ . It follows from Proposition 9.2 that the projection  $B(X, Y_i) \rightarrow BX$  is a covering with fiber  $Y_i$  whose covering

transformation group is the stabilizer  $\text{Stab}(y_i)$ . Thereby  $H_1(B(X, Y_i)) \cong \pi_1(B(X, Y_i))_{\text{ab}} \cong \text{Stab}(y_i)_{\text{ab}}$ . Finally, by considering all the connected components of  $B(X, Y)$ , we conclude

$$H_1^R(X, Y) \cong H_1(B(X, Y)) \cong \bigoplus_{i \in I} H_1(B(X, Y_i)) \cong \bigoplus_{i \in I} \text{Stab}(y_i)_{\text{ab}}. \quad \square$$

*Proof of Theorem 5.2.* We first show (37) below. Let  $Y = X$  be the primitive  $X$ -set. For each  $x_i \in X_i$ , we have  $e_{x_i} \in \text{Stab}(x_i)$  since  $x_i \triangleleft x_i = x_i$ . Hence the restriction of  $\epsilon_i : \text{As}(X) \rightarrow \mathbb{Z}$  on  $\text{Stab}(x_i)$  is also surjective, and permits a section  $\mathfrak{s} : \mathbb{Z} \rightarrow \text{Stab}(x_i)$  defined by  $\mathfrak{s}(n) = e_{x_i}^n$ . Here we remark that the action of  $\mathbb{Z}$  on  $\text{Stab}(x_i) \cap \text{Ker}(\epsilon_i)$  induced by the section is trivial. Indeed,  $g^{-1}e_{x_i}g = e_{x_i} \in \text{As}(X)$  for any  $g \in \text{Stab}(x_i)$  by (2). We therefore have  $\text{Stab}(x_i)_{\text{ab}} \cong (\text{Stab}(x_i) \cap \text{Ker}(\epsilon_i))_{\text{ab}} \oplus \mathbb{Z}$ . Hence it follows from Proposition B.5 that

$$H_1^R(X, X) \cong \bigoplus_{i \in \text{O}(X)} \text{Stab}(x_i)_{\text{ab}} \cong \mathbb{Z}^{\text{O}(X)} \oplus \bigoplus_{i \in \text{O}(X)} (\text{Stab}(x_i) \cap \text{Ker}(\epsilon_i))_{\text{ab}}. \quad (37)$$

We will show that  $H_2^Q(X)$  is isomorphic to the last summand. Recall  $H_2^R(X) \cong H_1^R(X, X)$  in Remark 9.3. It is known [LN, Theorem 2.1] that  $H_2^R(X) \cong H_2^Q(X) \oplus \mathbb{Z}^{\text{O}(X)}$ , and that a basis of  $\mathbb{Z}^{\text{O}(X)}$  is represented by  $(x_i, x_i) \in C_2^R(X)$  for  $i \in \text{O}(X)$ . By comparing the basis with the isomorphisms in (37), we complete the proof.  $\square$

## C Proof of Theorem 6.1

To prove Theorem 6.1 in an ad hoc way, we now observe concretely the map  $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$  for  $n \leq 3$ . Let us recall the rack complex  $C_n^R(X)$  in §5 and the (non-homogenous) standard resolution  $C_n^{\text{gr}}(\text{As}(X))$  of  $\text{As}(X)$  [see e.g. [Bro, §I.5]]. The map  $c_*$  was described in terms of their complexes by Kabaya [Kab, §8.4]. Actually he considered homomorphisms  $c_n : C_n^R(X) \rightarrow C_n^{\text{gr}}(\text{As}(X))$  for  $n \leq 3$  defined by

$$c_1(x) = e_x.$$

$$c_2(x, y) = (e_x, e_y) - (e_y, e_{x \triangleleft y}).$$

$$c_3(x, y, z) = (e_x, e_y, e_z) - (e_x, e_z, e_{y \triangleleft z}) + (e_y, e_z, e_A) - (e_y, e_{x \triangleleft y}, e_z) + (e_z, e_{x \triangleleft z}, e_{y \triangleleft z}) - (e_z, e_{y \triangleleft z}, e_A),$$

where we denote  $(x \triangleleft y) \triangleleft z \in X$  by  $A$  for short. As is known the induced map coincides with the map above up to homotopy (see [Kab, §8.4]).

We will construct a chain homotopy between  $t \cdot c_n$  and zero, when  $X$  is connected and of type  $t$ . Define a homomorphism  $h_i : C_i^R(X) \rightarrow C_{i+1}^{\text{gr}}(\text{As}(X))$  by

$$h_1(x) = \sum_{1 \leq j \leq t-1} (e_x, e_x^j),$$

$$h_2(x, y) = \sum_{1 \leq j \leq t-1} (e_x, e_y, e_{x \triangleleft y}^j) - (e_x, e_x^j, e_y) - (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_y, e_y^j, e_y),$$

$$h_3(x, y, z) = \sum_{1 \leq j \leq t-1} (e_x, e_y, e_z, e_A^j) - (e_x, e_z, e_{y \triangleleft z}, e_A^j) - (e_x, e_y, e_{x \triangleleft y}, e_z) - (e_y, e_{x \triangleleft y}, e_z, e_A^j)$$

$$\begin{aligned}
& + (e_x, e_z, e_{x \triangleleft z}^j, e_{y \triangleleft z}) + (e_z, e_{x \triangleleft z}, e_{y \triangleleft z}, e_A^j) + (e_x, e_x^j, e_y, e_z) - (e_x, e_x^j, e_z, e_{y \triangleleft z}) \\
& + (e_y, e_z, e_A, e_A^j) - (e_z, e_{y \triangleleft z}, e_A, e_A^j) - (e_z, e_{x \triangleleft z}, e_{x \triangleleft z}^j, e_{y \triangleleft z}) + (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j, e_z).
\end{aligned}$$

**Lemma C.1.** *Let  $X$  be as above. Then we have the equality  $h_1 \circ \partial_2^R - \partial_3^{\text{gr}} \circ h_2 = t \cdot c_2$ .*

*Proof.* Compute the two terms  $h_1 \circ \partial_2^R$  and  $\partial_3^{\text{gr}} \circ h_2$  in the left side:

$$\begin{aligned}
h_1 \circ \partial_2^R(x, y) &= \sum (e_x, e_x^j) - (e_{x \triangleleft y}, e_{x \triangleleft y}^j). \\
\partial_3^{\text{gr}} \circ h_2(x, y) &= \partial_3^{\text{gr}} \left( \sum (e_x, e_y, e_{x \triangleleft y}^j) - (e_x, e_x^j, e_y) - (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_y, e_y^j, e_y) \right) \\
&= \left( \sum (e_y, e_{x \triangleleft y}^j) - (e_x e_y, e_{x \triangleleft y}^j) + (e_x, e_x^j e_y) - (e_x, e_y) - (e_x^j, e_y) + (e_x^{j+1}, e_y) - (e_x, e_x^j e_y) \right. \\
&\quad \left. + (e_x, e_x^j) - (e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_x e_y, e_{x \triangleleft y}^j) - (e_y, e_{x \triangleleft y}^{j+1}) + (e_y, e_{x \triangleleft y}) \right) + (e_y, e_y^t) - (e_y^t, e_y) \\
&= t \left( (e_y, e_{x \triangleleft y}) - (e_x, e_y) \right) + (e_x^t, e_y) - (e_y, e_{x \triangleleft y}^t) - (e_y^t, e_y) + (e_y, e_y^t) + h_1 \circ \partial_2^R(x, y) \\
&= -t \cdot c_2(x, y) + h_1 \circ \partial_2^R(x, y).
\end{aligned}$$

Here we use Lemma 4.1 for the last equality. Hence, the equalities complete the proof.  $\square$

**Lemma C.2.** *Let  $X$  be as above. The difference  $h_2 \circ \partial_3^R - \partial_4^{\text{gr}} \circ h_3$  is chain homotopic to  $t \cdot c_3$ .*

*Proof.* This is similarly proved by a direct calculation. For this, recalling the notation  $A = (x \triangleleft y) \triangleleft z$ , we remark two identities

$$e_z e_A = e_{x \triangleleft y} e_z, \quad e_{y \triangleleft z} e_A = e_{x \triangleleft z} e_{y \triangleleft z} \in \text{As}(X).$$

Using them, a tedious calculation can show that the difference  $(t \cdot c_3 - h_2 \circ \partial_3^R - \partial_4^{\text{gr}} \circ h_3)(x, y, z)$  is equal to

$$\begin{aligned}
& (e_y, e_z, e_A^t) - (e_x^t, e_y, e_z) + (e_x^t, e_z, e_{e \triangleleft z}) - (e_y, e_{x \triangleleft y}^t) \\
& + (e_z, e_{x \triangleleft z}^t, e_{y \triangleleft z}) - (e_z, e_{y \triangleleft z}, e_A^t) + \sum_{1 \leq j \leq t-1} (e_y, e_y^j, e_y) - (e_{y \triangleleft z}, e_{y \triangleleft z}^j, e_{y \triangleleft z}).
\end{aligned}$$

Note that this formula is independent of any  $x$ , since the identity  $(e_a)^t = (e_b)^t$  holds for any  $a, b \in X$  by Lemma 4.1. However, the map  $c_3(x, y, z)$  with  $x = y$  is zero by definition. Hence, the  $t \cdot c_3$  is null-homotopic as desired.  $\square$

*Proof of Theorem 6.1.* The  $t \cdot c_*$  are null-homotopic immediately by Lemmas C.1 and C.2.  $\square$

The proof was a brute computation in an algebraic way; However this should be easily shown by a topological method:

**Problem C.3.** Does the  $t$ -vanishing of the map  $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$  hold for any  $n \in \mathbb{N}$ ? Provide its topological proof.

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Faculty of Mathematics, Kyushu University, 744, Motoooka, Nishi-ku, Fukuoka, 819-0395, Japan

E-mail address: nosaka@math.kyushu-u.ac.jp